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SOME ELEMENTARY PROPERTIES OF ILL CONDITIONED MATRICES AND LINEAR EQUATIONS

N. S. MENDELSOHN, University of Manitoba and Canadian Armament Research and Development Establishment

Introduction. It is a well known phenomenon that for certain sets of linear equations that errors due to rounding off during the computation can lead to large errors in the final solution. There have recently appeared in the literature a number of papers devoted to this phenomenon, usually called "ill condition", mostly concerned with methods of keeping computational errors in control. The devices used are either special ways of solving the equations or of subjecting the equations to a process referred to as "preconditioning." This latter process is a device for replacing the system of equations by one mathematically equivalent but in which rounding off errors have a small effect on the final solution.

The purpose of this paper is to describe in a very elementary way with simple examples the workings of the above mentioned phenomena. Some theorems which set out sharply the basis of the trouble will be stated and proved.

General remarks. We begin with an illustrative numerical example: Consider the set of equations

$$\begin{aligned} x + y &= 1 \\ x + 1.01y &= 2. \end{aligned} \tag{1}$$

These have as a solution $y=100$, $x=-99$. The solution is very sensitive to the value of the coefficient of y in the second equation. For example, if 1.01 is replaced by 0.99, a change of 2%, the solution becomes $y=-100$, $x=101$, a tremendous change in the solution.

We can replace the set of equations (1) by a second set which is mathematically equivalent but which is quite insensitive to changes in the coefficients. For example, if one subtracts 97 times the first equation of (1) from 100 times the second and then subtracts 700 times the second from 704 times the first one obtains the equations,

$$\begin{aligned} 3x + 4y &= 103 \\ 4x + 3y &= -696. \end{aligned} \tag{2}$$

The set (2) is mathematically equivalent to the set (1) but is obviously less sensitive to small percent changes in the coefficients than in the set (1). One can say that replacing (1) by (2) is "preconditioning" the equations for computation. It must be clearly understood that preconditioning is only useful where the errors introduced by changes in the coefficients are at the control of the computer. For example, if the equations (1) are obtained from a physical experiment in which

all the coefficients are measured quantities and the coefficient 1.01, for instance, means a quantity between .99 and 1.03 then there is an essential indetermination in the numbers x and y which no amount of preconditioning of the equations can help. In this case, perhaps, all our abuse should be heaped upon the physical experiment rather than upon the equations. Where preconditioning is useful is in those cases where errors are introduced by the computing process itself. The most obvious example is the case where at each stage in the computation the numbers obtained are rounded off.

The phenomenon of "ill condition" is more usually ascribed to a matrix than to a set of linear equations, the matrix in the case of a set of linear equations being that of the coefficients of the variables. In [1] and [2] of our bibliography, various measures of "ill condition" are given. The one due to von Neumann and Goldstine, namely the ratio of moduli of the largest to the smallest of the characteristic roots, will be given an interesting justification in Theorem 3 of this paper. In passing, we note that the von Neumann and Goldstine condition number of the matrix of equations (1) in our example is about 400, while that of equations (2) is 1.

In general a system of linear equations with matrix A has a solution whose sensitivity to changes in the coefficients is roughly proportional to the elements of A^{-1}. More explicitly, if the equations are

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, 3, \cdots, n)$$

and $A = (a_{ij})$ with $A^{-1} = (c_{ij})$, then

$$\frac{\partial x_i}{\partial b_j} = c_{ij}, \qquad \frac{\partial x_i}{\partial a_{jk}} = -c_{ij}x_k.$$

Roughly speaking, an ill-conditioned matrix has for its inverse a matrix whose elements are relatively large. In the work that follows we will show that the computation of an *approximate* inverse to an ill-conditioned matrix gives rise to a complication, namely, that it is necessary to distinguish between an *approximate left inverse* and an *approximate right inverse*. An example will be given to show that if linear equations are solved by matrix inversion, ill-condition is not as serious as might first appear provided one is careful not to confound approximate left inversion with approximate right inversion.

Theoretical development. We first prove a theorem to show that there are matrices whose determinants are arbitrarily sensitive to changes in one of the elements:

THEOREM 1. *Let ϵ and K be two arbitrary positive real numbers. For each $n \geqq 2$, there exists a non-singular $(n \times n)$ matrix with the property that multiplication of a specified one of its non-zero elements by $1+\epsilon$ will multiply its determinant by $1+K$.*

Proof. Let the matrix be $A=(a_{ij})$ $(i, j=1, 2, \cdots, n)$; let the specified element be a_{11} and let A_{ji} be the cofactor of a_{ij}. We are to choose A so that the following equation amongst determinants holds:

$$\begin{vmatrix} a_{11}(1+\epsilon) & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} = (1+K) \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}.$$

This is equivalent to $\epsilon a_{11}A_{11}=K(a_{11}A_{11}+U)$ where $U=a_{12}A_{21}+a_{13}A_{31}+\cdots+a_{1n}A_{n1}$. Here neither A_{11} nor U involves the element a_{11}. The value of a_{11} is given by

$$a_{11} = \frac{-KU}{A_{11}(K-\epsilon)}. \tag{3}$$

For $K \neq \epsilon$, we need only choose the elements of A (except for a_{11}) in such a way that U and A_{11} are distinct from 0 and then choose a_{11} by equation (3). If $K=\epsilon$ the theorem is true for any matrix A of the form

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots 0 \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn} \end{pmatrix}$$

provided A has a non-zero determinant.

THEOREM 2. *Let ϵ and K be two arbitrary positive real numbers. For each $n \geqq 2$ there exist non-singular $(n \times n)$ matrices A and X such that XA has each of its elements differing from the corresponding element of the identity matrix by less than ϵ and such that AX has all of its elements greater in absolute value than K.*

Proof. We will prove the 2×2 case here leaving the general proof to the end of the paper. This is done because our next theorem concerns only 2×2 matrices and is a companion theorem to this one.

Let ϵ^* be any number such that $0<\epsilon^*<\min(1, \epsilon)$ and let K^* be a number such that $K^*>\max(K, 2)$.

Let

$$A = \begin{pmatrix} \dfrac{K^*}{\epsilon^*} & 1 \\ \dfrac{K^{*2}}{\epsilon^{*2}} - 1 & \dfrac{K^*}{\epsilon^*} \end{pmatrix}$$

and let

$$X = \begin{pmatrix} K^* + \dfrac{K^* + K^{*2}}{\epsilon^*} - \epsilon^* & -1 - K^* - \epsilon^* \\ K^* + \dfrac{K^{*2}}{\epsilon^*} - \dfrac{K^{*2}}{\epsilon^*} - \epsilon^* + 1 & \dfrac{K^*}{\epsilon^*} - \epsilon^* - K^* \end{pmatrix}.$$

By direct computation,

$$XA = \begin{pmatrix} 1 + \epsilon^* & -\epsilon^* \\ \epsilon^* & 1 - \epsilon^* \end{pmatrix},$$

so that XA has its elements differing from those of the identity matrix by less than ϵ.

Also by direct computation

$$AX = \begin{pmatrix} \dfrac{K^{*3}}{\epsilon^{*2}} + \dfrac{2K^{*2}}{\epsilon^*} + 1 - \epsilon^* & -\left(\dfrac{K^{*2}}{\epsilon^*} + 2K^* + \epsilon^*\right) \\ \epsilon^*\left(\dfrac{K^{*2}}{\epsilon^{*2}} + \dfrac{K^*}{\epsilon^*} - 1\right)^2 & -\left(\dfrac{K^{*2}}{\epsilon^{*2}} + \dfrac{2K^{*2}}{\epsilon^*} - 1 - \epsilon^*\right) \end{pmatrix}.$$

Our choice of K^* and ϵ^* assures that the magnitude of every element of AX is greater than K.

The phenomenon exhibited in Theorem 2 is always associated with ill-conditioned matrices. In the next theorem we will show that the von Neumann and Goldstine condition number for such matrices is large (*i.e.* for small ϵ and large K). We will normalize the problem so that the characteristic roots have a product equal to unity in which case we may replace the ratio of the characteristic roots by the difference.

THEOREM 3. *Let ϵ and K be arbitrary positive numbers such that $K/\epsilon > 4$. Let A and X be two (2×2) matrices with the properties*:

(1) *A has non-negative elements and* $\det A = 1$.

(2) *XA has all its elements differing from those of the identity matrix by ϵ.*

(3) *AX has all its elements greater in magnitude than K.*

If x_1 and x_2 are the characteristic roots of A, then

$$|x_1 - x_2| > \sqrt{\frac{K}{\epsilon} - 4}.$$

Proof. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

a, b, c, d are all non-negative and $ad-bc=1$. We are given that

$$XA = \begin{pmatrix} 1 \pm \epsilon & \pm\epsilon \\ \pm\epsilon & 1 \pm \epsilon \end{pmatrix}.$$

The proof is similar for all changes of signs so we will consider only the case where

$$XA = T = \begin{pmatrix} 1+\epsilon & -\epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}.$$

Then $X = TA^{-1}$, so that

$$AX = ATA^{-1} = \begin{pmatrix} 1+\epsilon(a+b)(c+d) & -\epsilon(a+b)^2 \\ \epsilon(c+d)^2 & 1-\epsilon(a+b)(c+d) \end{pmatrix}.$$

By hypothesis;

$$\epsilon(a+b)^2 > K$$
$$\epsilon(c+d)^2 > K$$
$$1+\epsilon(a+b)(c+d) > K$$
$$\epsilon(a+b)(c+d) - 1 > K.$$

From the first two of these inequalities

$$a+b > \sqrt{\frac{K}{\epsilon}}, \qquad c+d > \sqrt{\frac{K}{\epsilon}}.$$

Using $ad-bc=1$, the characteristic roots of A satisfy $x^2-(a+d)x+1=0$. This has roots x_1 and x_2 such that $|x_1-x_2| = \sqrt{(a+d)^2-4}$. We also have $c > \sqrt{K/\epsilon}-d$ and $b > \sqrt{K/\epsilon}-a$.

If both $\sqrt{K/\epsilon}-a$, $\sqrt{K/\epsilon}-d$ are positive we have

$$bc = \left(\sqrt{\frac{K}{\epsilon}} - a\right)\left(\sqrt{\frac{K}{\epsilon}} - d\right)$$

or

$$(a+d)\sqrt{\frac{K}{\epsilon}} > \frac{K}{\epsilon} + ad - bc$$

or

$$(a+d)\sqrt{\frac{K}{\epsilon}} > \frac{K}{\epsilon} + 1.$$

Hence

$$a + d > \sqrt{\frac{K}{\epsilon}} + \sqrt{\frac{\epsilon}{K}}$$

or

$$(a + d)^2 > \frac{K}{\epsilon} + \frac{\epsilon}{K} + 2.$$

Hence

$$(a + d)^2 - 4 > \frac{K}{\epsilon} + \frac{\epsilon}{K} - 2,$$

and

$$|x_1 - x_2| > \sqrt{\frac{K}{\epsilon} + \frac{\epsilon}{K} - 2} > \sqrt{\frac{K}{\epsilon} - 4}.$$

If at least one of $\sqrt{K/\epsilon}-d$, $\sqrt{K/\epsilon}-a$ is negative, suppose $\sqrt{K/\epsilon}-a$ is negative. Then $a^2>(K/\epsilon)$. Hence, $(a+d)^2-4>a^2-4>(K/\epsilon)-4$ and, $|x_1-x_2|>\sqrt{K/\epsilon-4}$.

The author has proved a similar theorem under weakened hypotheses *e.g.*, "XA has all of its elements differing from those of the identity matrix by ϵ," may be replaced by "XA has one of its elements differing from the corresponding element of the identity matrix by ϵ while the remaining elements of XA differ in absolute value from the corresponding elements of the identity matrix by an amount less than or equal to ϵ." Also the statement "AX has all its elements greater in magnitude than K" may be replaced by "AX has at least one of its elements greater in magnitude than K." The proofs are considerably more complicated and would not add anything to the point of our discussion.

Before giving a numerical application we may note the following consequence of Theorem 2. For a specified degree of accuracy an ill-conditioned matrix A may be associated with matrices X and Y where XA and AY differ from I by at most the specified error while AX and YA may differ considerably from I. We may then say that X is a suitable left inverse but not a suitable right inverse of A while Y is a suitable right inverse but an unsuitable left inverse of A. If one is to solve a system of linear equations by matrix inversion it seems reasonable that one should not confuse X and Y. More specifically, consider the set of equations

$$\sum_{j=1}^{n} a_{ij}u_j = b_i \qquad (i = 1, 2, 3 \cdots n).$$

As a matrix equation this may be written

$$AU = B \quad \text{where} \quad A = (a_{ij}), \quad \text{and}$$

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{pmatrix}.$$

Here a solution is $U=A^{-1}B$. If we are approximating to A^{-1} it is obvious that what is required is an approximate *left* inverse of A. On the other hand if we consider the equations

$$\sum_{i=1}^{n} a_{ij}v_i = c_j \qquad (j = 1, 2, \cdots n)$$

and write them in matrix form $VA=C$, where $A=(a_{ij})$, $V=(v_1, v_2, \cdots, v_n)$, $C=(c_1, c_2, \cdots, c_n)$, the solution $V=CA^{-1}$ requires that A^{-1} be approximated by an approximate right inverse of A. A numerical example which follows illustrates the error introduced by confusing an approximate left inverse with an approximate right inverse.

Numerical example. Consider the set of equations

$$\begin{aligned} x + y + z &= 6.00 \\ x + y + 1.01z &= 6.03 \\ x + 0.99y + z &= 5.98. \end{aligned} \tag{4}$$

These equations have an exact solution $x=1$, $y=2$, $z=3$. Equations (4) can be written in the matrix form:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1.01 \\ 1 & 0.99 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6.00 \\ 6.03 \\ 5.98 \end{pmatrix}. \tag{5}$$

The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1.01 \\ 1 & 0.99 & 1 \end{pmatrix}$$

is easily seen to be an ill-conditioned matrix. If we do not invert exactly it appears that an approximate left inverse is more appropriate than an approximate right inverse. We will assume computations are carried out in some systematic manner and rounded off to two decimal places. Conceivably, one might compute as an approximate left inverse the matrix

$$X = \begin{pmatrix} 1.01 & -100 & 100 \\ 99.99 & 2 & -102 \\ -162.01 & 102 & 0 \end{pmatrix}.$$

Here

$$XA = \begin{pmatrix} 1.01 & .01 & .01 \\ -.01 & 1.01 & .01 \\ -.01 & -.01 & 1.01 \end{pmatrix}$$

while

$$AX = \begin{pmatrix} -2.01 & 4 & -2 \\ -2.03 & 5.02 & -2 \\ -2.01 & 3.98 & -0.98 \end{pmatrix}.$$

Hence X is a good approximation as a left inverse of A but a poor approximation as a right inverse. If we use X to solve (5) we obtain

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X \begin{pmatrix} 6.00 \\ 6.03 \\ 5.98 \end{pmatrix}$$

which yields $x=1.06$, $y=2.04$, $z=3.00$, which is a good approximation to the exact solution considering the condition of A.

In the same way the matrix

$$Y = \begin{pmatrix} 1.01 & -101.99 & 100.01 \\ 102 & 2 & -100 \\ 102 & 100 & 0 \end{pmatrix}$$

has the properties

$$AY = \begin{pmatrix} 1.01 & .01 & .01 \\ -.01 & 1.01 & .01 \\ -.01 & -.01 & 1.01 \end{pmatrix}$$

and

$$YA = \begin{pmatrix} -0.97 & -1.97 & -1.99 \\ 4 & 5 & 4.02 \\ -2 & -2 & -1 \end{pmatrix}$$

so that Y is a good approximate right inverse but a poor approximate left inverse.

If one ignores the distinction between an approximate left inverse and an approximate right inverse and tries to use Y to solve (5) in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = Y \begin{pmatrix} 6.00 \\ 6.03 \\ 5.98 \end{pmatrix},$$

one obtains $x=-10.88$, $y=26.06$, $z=-9$ an answer with a fantastically large error.

Since, for computing purposes, distinction between left and right approximate inverse is of some importance (and of utmost importance in the case of very badly conditioned matrices) it is worthwhile to look into some of the methods of matrix inversion to determine what kind of inverse they determine when used approximately. For example; Albert in [5], gives a method which when used approximately gives a left inverse while Andree in [6] gives a method which approximates to a "hybrid" inverse. It is not easy to see in most cases whether the method of approximation leads to a one-sided or hybrid inverse. Other things being equal a one-sided inverse is preferable to a hybrid. Where iterative processes are used it is advisable to calculate as a left inverse and check each stage for accuracy as a right inverse. In the case of symmetric matrices any method of inverse approximation which yields a symmetric matrix as an inverse will be good since the resultant matrix has the same error both as a left and as a right inverse. These remarks are mainly of use because one cannot tell in many cases whether or not a matrix is ill-conditioned. Many innocent looking matrices are very badly conditioned. Finally, where a set of equations has an obviously ill conditioned matrix and preconditioning can easily be carried out, it is advisable to do so.

We close with a proof of Theorem 2 in the general case.

Proof of Theorem 2 for $(n\times n)$ matrices. Let ϵ^* be any number such that $0<\epsilon^*<\min(1, \epsilon)$. Let $A=(a_{ij})$ $(i, j=1, 2, \cdots, n)$, where a_{ij} are to be determined later in such a way that $\det A=1$.

Let $A^{-1}=A_{ij}$ $(i, j=1, 2, \cdots, n)$. Define X by the equation

$$X=\begin{pmatrix} 1+\epsilon^*, & \epsilon^*, & \cdots, & \epsilon^* \\ \epsilon^*, & 1+\epsilon^*, & \cdots, & \epsilon^* \\ \vdots & & & \\ \epsilon^* & \epsilon^*, & \cdots, & 1+\epsilon^* \end{pmatrix} A^{-1}.$$

Then $X=(I+E)A^{-1}$ where I is the identity matrix and

$$E=\begin{pmatrix} \epsilon^*, & \epsilon^*, \cdots, & \epsilon^* \\ \epsilon^*, & \epsilon^*, \cdots, & \epsilon^* \\ \vdots & & \\ \epsilon^*, & \epsilon^*, \cdots, & \epsilon^* \end{pmatrix}.$$

Denote E by (l_{ij}) where $l_{ij}=\epsilon^*$ $(i, j=1, 2, \cdots, n)$. It follows that $AX=I+AEA^{-1}=I+Y$ where $Y=AEA^{-1}$. If $Y=(y_{ij})$ $(i, j=1, 2, \cdots, n)$ we have

$$y_{ij}=\sum_{u=1}^{n}\sum_{v=1}^{n} a_{iu}l_{uv}A_{vj}=\epsilon^*\left(\sum_{m=1}^{n} a_{im}\right)\left(\sum_{r=1}^{n} A_{rj}\right).$$

Suppose we can choose A so that

(1) $\det A = 1$.

(2) The sum of the elements in each row of A is greater than $2K/\epsilon^*$.

(3) All sums $\sum_{r=1}^{n} A_{rj}$ are non-zero integers.

Then all the y_{ij} are greater in magnitude than $2K$, so that all the elements of AX are greater in magnitude than $2K-1$. Let K^* be a positive integer satisfying the conditions $K^*>2$ and $K^*>2K/\epsilon^*$ and put

$$B = \begin{bmatrix} 1 & K^* & K^* \cdots K^* & K^* \\ 0 & 1 & K^* \cdots K & K^* \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & 0 \cdots 1 & K^* \\ 0 & 0 & 0 \cdots 1 & 1+K^* \end{bmatrix}.$$

Let the inverse of B have elements B_{ij} $(i, j=1, 2, 3, \cdots, n)$. Then B has the properties (1) and (2). With regard to property (3) we can say that all sums $\sum_{r=1}^{n} B_{rs}$ are integers at least one of which is different from 0. For if all $\sum_{r=1}^{n} B_{rs}$ were equal to 0, suppose we choose Z so that $ZB=I+E$. Then $BZ=I+BEB^{-1}$. We have $BEB^{-1}=0$ by the condition $\sum_{r=1}^{n} B_{rs}=0$ for all s. Hence Z is an exact right inverse of B, a contradiction, since $ZB=I+E$ and $E\neq 0$. The matrices B and Z then have the properties:

(a) Each element of ZB differs from the corresponding element of the identity matrix by an amount less than ϵ.

(b) Not all elements which are off the main diagonal of BZ are equal to 0, and each such non-zero element of BZ is greater in magnitude than K.

If BZ has no element equal to 0, it is also true that every element, including those on the main diagonal, is greater in magnitude than K. In this case B satisfies the conditions imposed on A by the theorem.

If BZ has at least one zero we proceed as follows: Let $C_2, C_3, \cdots, C_n$ be indeterminates to be replaced by positive integers. Form a matrix A as follows: the first row of A is taken to be the same as the first row of B. For $(i=2, 3, \cdots, n)$, $(j=1, 2, 3, \cdots, n)$, take $a_{ij}=b_{ij}+C_i b_{ij}$. Then A satisfies the required conditions (1) and (2). With regard to the condition (3), each sum $\sum_{r=1}^{n} A_{rs}$ is a linear combination of the B_{ij} with coefficients products of the C_i and such that at least one of the B_{ij} in the sum is different from 0. Hence, for each s, $\sum_{r=1}^{n} A_{rs}$ is a polynomial in $C_2, C_3, \cdots, C_n$ which is not identically 0. As such a polynomial equated to 0 has at most a finite number of solutions in the C_i, there exist choices of $C_2, C_3, \cdots, C_n$ such that each C_i is a positive integer and such that for each s, $\sum_{r=1}^{n} A_{rs}\neq 0$. For such a choice the matrix A satisfies the conditions of the theorem.

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CONSISTENCY AND COMPLETENESS—A RÉSUMÉ

FRANK DESUA, Ohio University

The modern foundational investigations of mathematics, encroaching as they do on unresolved philosophical questions, have had their polemic aspects. Controversy over ontological problems gave rise at the turn of the century to the three "schools" associated with the programs of Russell and Whitehead, of Hilbert, and of Brouwer, a trichotomy which is preserved to some extent even today. However, those results of the investigations which are perhaps the most significant for the mathematician, namely those embodied in the famous incompleteness theorems of Gödel, must be considered as well established and unimpeachable as any found in the whole of mathematics, based as they are on methods acceptable to the devotees of each of these programs.

Among the values to be gained from these investigations is the insight afforded into the nature of our mathematical preconceptions, and the revelation that even some of the most compelling of these may actually be misconceptions. Of course, once this is discovered, the void created by the dissolution of these misconceptions is apt to be immediately filled by newly formed assumptions, consciously or otherwise. The history of mathematics, as of any science, is to some extent the story of the continual replacement of one set of misconceptions by another. This is of course no cause for despair, for the newly instated assumptions very often possess the merit of being closer approximations to truth than those which they replace.

We are concerned in this paper with the detection of one such misconception

—a misconception concerning the scope and power of formal mathematical methods—and Gödel's remarkable discovery of largely unsuspected limitations inherent in such methods. This discovery is part of, and a product of, the "crisis in the foundations" of which the rumblings have not yet died away. It is not yet entirely clear what new assumptions are to supplant the old, but it is certainly clear that the old must give way.

Our paper is divided into three sections, the first of which is entitled "the Problem," which is the problem of the consistency and completeness of mathematics, first explicitly formulated by Hilbert. The last section "the Solution," concerns the surprisingly negative solution of Gödel. The second section, "Formal Systems," is a digression necessary for an appreciation of the nature of the solution. Before we attempt a statement of the problem, we concern ourselves to some extent with how it arose.

The problem. The theory of abstract sets or classes may be said to be the basic discipline of pure mathematics. Belief in the existence of sets is almost universal among mathematicians, a belief which seems to be preordained by the dictates of common sense. So compelling is the notion of set that even those who harbor doubts must at least feign belief part of the time in order to discourse intelligibly. Yet this concept, despite its seeming necessity and superficial innocence, has proved to be the Pandora's Box of mathematics.

Consider how sets are specified. Two methods are conceivable. A set may be specified *extensionally, i.e.*, by a listing of the members such as "the set consisting of the numbers 2, 3, 5, and 7" or *intensionally* through some defining property or necessary and sufficient condition for membership—technically known as a "monadic propositional function"—as "the set of all primes less than 10." The extensional specification of sets is, of course, precluded as soon as we consider infinite sets.

The intensional method is often canonized in the form of a so-called logical principle, the (unrestricted) Principle of Abstraction, which states that every property determines a set, or in technical language, every monadic propositional function has exactly one set as extension. Symbolically, this becomes:

$$(\exists S)(x)[x \in S \leftrightarrow P(x)]$$

where S is the set determined and P is the defining property.

Briefly the argument in support of this principle is this: Every condition or property determines the set of all entities possessing the property or obeying the condition (even in the case where no such entity exists, the set determined being empty). Conversely every set determines at least one property, namely the property of being a member of the set, if no other.

The principle is widely appealed to in mathematics—often in some restricted form—either explicitly or tacitly. It is implicit, for example, in the high school geometry definition of a "locus" as a point set every element of which obeys some metric condition.

In the principle of abstraction, then, we have an intuitively obvious prin-

ciple of wide applicability. Unfortunately there is one thing wrong with it. In the unrestricted form stated above, it happens to be false. It was proved to be false by Bertrand Russell in 1901.

Let's examine the process of set formation as embodied in the principle of abstraction. Out of the aggregate of all entities we choose those which possess a given property to form our set. The flaw in the principle of abstraction lies in the fact that it ignores one entity, namely the newly formed set. Conceivably the property used to form the set may neither apply nor fail to apply to this set. In this case no set is actually determined. Russell discovered a property, possessed by many entities, yet which cannot consistently be said to determine any set.

The property in question is the property which a set x may possess of "not being a member of itself" ($x \notin x$). Many sets possess the property—a set of material objects, for example—while other sets fail to possess the property—the set of all sets would, assuming the existence of such a set. Indeed in an unrestricted universe of discourse (assuming the allowability of such a universe), there must be as many sets of one kind as of the other, for the complement of any set possessing the property is a set which does not, and conversely.

It is easily seen that the above mentioned property determines no set, for suppose there were such a set S. From the defining property, $x \in S \leftrightarrow x \notin x$. Does S itself possess the property, or not? Putting S for x we have, $S \in S \leftrightarrow S \notin S$, which is a contradiction. Therefore no such set S exists, although innumerable sets x possess the property $x \notin x$.

Actually, the first proof of the non-validity of the principle of abstraction came several years earlier in a result by Burali-Forti concerning ordinal numbers. Sets of numbers of various kinds are common objects of study in mathematics. It would seem then that the set of all ordinals is perfectly well defined according to the canons of common sense mathematics. Given any entity we can decide whether it is a member of the set or not if we can determine whether it is an ordinal number or not. It came therefore as a surprise when in 1897 Burali-Forti demonstrated the non-existence of the set of all ordinals.

To recapitulate the proof, recall the following theorems of ordinal number theory: Any initial segment of the ordinals, that is, any set of ordinals containing all ordinals preceding any of its members, can be ordered according to the magnitude of its members. The resulting ordered set is well-ordered; consequently it has an ordinal number. Further this number exceeds any member of the set (in fact, it is the least number which exceeds every member of the set).

Now suppose the property of being an ordinal number determined a set S, *i.e.*, let S be the set of all ordinals. Then S would be a set to which the above theorems apply, so that it would have an ordinal number Ω. Accordingly, Ω would exceed every member of S. But if S is the set of all ordinals, Ω must be one of its members. Consequently, Ω exceeds itself, which is a contradiction.

Superficially somewhat similar difficulties arise, not only in the theory of sets, but also in the theory of statements or propositions. It would seem that

any statement which purports to be a statement of fact must be either true or false. If the statement conforms with the facts it is true, otherwise false. Thus, for example, the statement "John has quit smoking" is true only if John has actually quit smoking. (There are difficulties of an existential nature in some cases, as for example with such statements as "Pegasus has wings" or "Four sided triangles are isosceles." But such difficulties are capable of resolution through the adoption of certain conventions of interpretation, as in Russell's and Quine's theories of singular terms and descriptions.)

This view of the validity of statements, however, is subject to difficulties of another kind if no restrictions are imposed on statement formation. For if we are talking about statements themselves, *i.e.*, if we consider statements about statements, as we often do in mathematics, the statements themselves become part of the facts. Conceivably in an unrestricted discussion the facts may be so affected as to render certain such statements about statements neither true nor false.

That this is actually the case was discovered by the ancient Greeks. They constructed a statement (the Epimenides), of the form "This statement is false," which asserts its own falsehood. This statement, referring as it does to itself, can only be true if it is indeed false, and conversely.

It might be pointed out that such Epimenidean statements cannot be constructed in any properly restricted mathematical system, so that the contradiction does not arise in practice. (The Russell and Burali-Forti contradictions actually did arise in mathematical discourse, and still do occasionally.) However, as we shall see, something very closely resembling such Epimenidean statements actually can be constructed in certain formal systems, namely Gödelian formulas. This is a profound syntactical fact which we shall examine later.

A great many of these demonstrations running counter to intuitive principles have unexpectedly come to light. Cantor proved the non-existence of the set of all cardinal numbers. Berry and Richard have shown the non-existence of certain apparently well defined denumerable sets of finite numbers. Skolem has shown conversely the existence of certain number sets possessing no defining property.

We have taken the position here that these "contradictions" actually constitute demonstrations, by *reductio ad absurdum*, of the non-validity of certain intuitive principles of long standing, such as the principle of abstraction. Other positions concerning the significance of these contradictions are widely held. The authors of the contradictions themselves very often tacitly defended these principles, and consequently regarded the contradictions as "paradoxes." They had then to revise their concept of "property" or "condition," rather than "set," *etc.* Whatever position is taken however, it is clear that unbridled intuitive reasoning, even on the most basic level, can and sometimes does lead to difficulties.

With this realization we are now in a position to state what was until recently a central problem of the modern foundational investigations of mathematics, namely to devise restrictions on intuitive reasoning sufficiently strong

to preclude the possibility of inconsistency but sufficiently weak to allow mathematics the greatest possible latitude for development, at least to allow the development it has already undergone. Further the problem is to prove, if possible, by unimpeachable methods (and therein lies a danger—the principles discredited by Russell were "unimpeachable" in their day) that these restrictions are actually sufficiently strong, *i.e.*, that the resulting mathematical systems are consistent.

With the first part of this problem we shall not concern ourselves except to say that restrictions have been devised which seem to fulfill the requirements. There are Russell's type theory, Ramsey's simplification of this theory, Quine's stratification, and Zermelo's axiomatic set theory to mention a few. The *adequacy* of the resulting system for the derivation of classical mathematics is proved very directly. Either classical mathematics is actually derived within the system (as in Russell and Whitehead's *Principia Mathematica*) or it is shown that any theorem that can be derived in, say, *Principia*, can also be derived in the system under consideration. The proof of the consistency of these systems is quite another matter, and this is the part of the problem with which we are mainly concerned.

Formal systems. The systems whose consistency is investigated are not actual systems of ordinary intuitive mathematics but rather certain specializations thereof known as formal systems or formal languages. We shall investigate this concept in some detail presently, but here we want to point out differences between formal mathematics and the postulate systems of ordinary mathematics. In ordinary mathematics it is helpful in carrying out deductions to know the meanings of the technical terms (or "undefined" terms) involved. The semantics of the system is of consequence. Even when the system is semi-formalized with regard to technical terms, so that they become truly undefined, semantics plays a heavy role in the logic involved in the deductions, which is usually on a primitive verbal level.

In formal mathematics on the other hand the technical terms are analyzed to the point where all their properties which matter for the deduction of theorems have been expressed by axioms. The terms must then have a recognizable form rather than an apprehensible meaning. It is then possible to treat these terms as words essentially without meaning, for to say that these terms have meanings essential to the deductions amounts to saying that not all their properties have been expressed by axioms, *i.e.*, that the analysis is incomplete. Even the logic of the system is explicitly incorporated into the system in the form of axioms and rules of inference, and the whole is rendered in symbolic rather than verbal form. When this analysis is completed, and meanings thus left out of account, questions of form or structure become the sole consideration. Semantics has been replaced by syntax.

There are several good reasons for this preference for formal over intuitive systems in many foundational investigations, not the least of which is the fact that intuitive mathematics is a function of the mathematician as well as of the

subject matter. Whereas a topologist may embrace such principles as the axiom of choice or the continuum hypothesis without hesitancy, an algebraist may do so with compunction. Thus it becomes of interest to compare systems in which these principles are explicitly included with systems in which they are excluded. Also standards of rigor vary with the times and the mathematician, one man's proof being another man's sophistry. The intuitive mathematics of the great eighteenth century analysts, for example, is demonstrably inconsistent. Questions of rigor do not arise however in a system restricted to finitary and constructive methods of proof as is the case in many formal systems. Perhaps the chief reason for the use of formal systems rather than intuitive systems, however, is that only within formal systems can such vague semantical concepts as *axiom*, *theorem*, *proof*, *consistency*, *etc.*, be made explicit and precise. Formalization is a necessary concomitant to precision.

The study of formal systems has been criticized (and extolled!) on the grounds that it "reduces mathematics to a meaningless game." It seems plausible that such criticism is based on a misconception concerning the purpose of these studies. It would be as much to the point to criticize the study of topological spaces on the grounds that such "spaces" are incapable of propagating electromagnetic radiation. It must be kept in mind that the partial formalization of much of mathematics—of algebra, for example—has proceeded steadily since the time of the Babylonians, yet algebra certainly remains meaningful mathematics. Abstraction of meaning does not imply loss of meaning.

The extent to which intuitive mathematics can be formalized is a problem on which much light has been shed by recent foundational investigations. That at least a very considerable measure of formalization is possible for mathematical theories is evident upon a perusal of some of the attempts at such formalization—Whitehead and Russell's *Principia Mathematica*, for example. But that limitations do exist on the kinds of problems which can be stated and proved within a formal system is revealed by the researches of Gödel, Church, and others. And the delineation of such limitations is the subject of continuing research. Involved in the process of formalizing a theory may be difficulties as great as those involved in the subject matter of the theory itself.

Let's examine the way in which a formal system is actually set up. We start with a certain set of symbols which may be infinite in number. Any finite linear array of these, allowing repetitions, is called a *formula*. The formulas are divided into two disjoint sets called *well-formed formulas* (w.f.f.) and *gibberish* (only the w.f.f. are to have meaning in a semantic *interpretation* of the system). The w.f.f. are defined recursively, and in a purely syntactical or morphological manner, *i.e.*, solely in terms of their structure. Moreover they are defined effectively, *i.e.*, a finite process is specified whereby we can determine whether a given formula is well-formed or not.

A certain subset of the w.f.f. are chosen to be our initial *axioms*. (The age-old criterion of "self-evidence" of axioms is dispensed with, there being no de-

grees of perspicuity in a system from which all meaning has been abstracted.) New w.f.f. are obtained or "derived" from these axioms by the application of certain *rules of inference*, which again are recursive, syntactical, and effective. By a *proof* we mean a finite sequence of w.f.f.'s, each of which is either an axiom or is obtainable from one or more earlier formulas of the sequence by an application of a rule of inference.

The last formula of a proof is, by definition, a *theorem*. We write $\vdash A$ to denote that A is a theorem. (The symbol '$\vdash$' is called the assertion symbol.) Whether a given sequence of formulas constitutes a proof or not can be effectively determined, but whether a given formula is a theorem or not generally can not be determined by any effective uniform procedure.

An actual example of a formal system may serve to clarify some of these definitions. The following is one possible formalization of a fragment of modern logic known as the "propositional calculus":

Symbols: An infinite list of lower case letters, primed and unprimed: $a, b, c, \cdots, a', b', c', \cdots, a'', b'', c'', \cdots$
parentheses: (, and)
negation symbol (tilde): $\sim$
alternation symbol: $\vee$
Any finite array of these, for example $\vee))a'\sim$, is a formula. This particular one happens to be gibberish.

w.f.f.: Any letter (primed or unprimed) alone is w.f. If A and B are w.f., so are $\sim A$ and $(A \vee B)$. Examples: r is w.f. So are $\sim r$ and $(r \vee \sim r)$.

Axioms: An infinite list of axioms which come under the following axiom schemata:
(1) $(A \vee \sim(A \vee A))$
(2) $((\sim B \vee C) \vee B)$
(3) $(\sim(D \vee \sim E) \vee (\sim(E \vee F) \vee (F \vee D)))$
Each of these schemata represents an infinite set of axioms, namely one for each possible choice of w.f.f. in place of A, B, C, D, E, and F.

Rule of inference: From two formulas of the form G and $(\sim G \vee H)$, we infer the formula H.

This system is capable of a semantical interpretation in terms of the so-called logical connectives "not," "either · · · or · · · ," *etc.* The letters may stand for sentences or propositions. Thus if we let "r" denote "it is raining," then "$\sim r$" becomes "it is not raining" and "$(r \vee \sim r)$" becomes "either it is raining or it is not." As an example of a theorem and its proof we assert that $\vdash (r \vee \sim r)$.

Proof:

(i) $(\sim r \vee \sim(\sim r \vee \sim r))$
[from axiom schemata 1 replacing A by $\sim r$]

(ii) $(\sim(\sim r \vee \sim(\sim r \vee \sim r)) \vee (\sim((\sim r \vee \sim r) \vee r) \vee (r \vee \sim r)))$
[from 3 replacing D by $\sim r$, E by $(\sim r \vee \sim r)$, and F by r]

(iii) $(\sim((\sim r \vee \sim r) \vee r) \vee (r \vee \sim r))$
[using the rule of inference on steps (i) and (ii)]
(iv) $((\sim r \vee \sim r) \vee r)$
[from 2 replacing B by r, and C by $\sim r$]
(v) $(r \vee \sim r)$
[using the rule of inference on steps (iii) and (iv)]

Thus we have "proved" that it is either raining or it is not.

We note here, for later use, a binary connective of considerable importance, namely the *conditional* or *material implication*, symbolized by "$\rightarrow$," and translated by one sense of the idiom "if $\cdots$ then $\cdots$." If A and B are w.f., then $(A \rightarrow B)$ is w.f. A rule of inference equivalent to that given in the above system, but much more widely used, is *modus ponens* (or strictly speaking, its syntactical analogue, *detachment*) which allows us to obtain from two formulas of the form A and $(A \rightarrow B)$ the formula B.

Two further definitions are important for our discussion. A formal system is *consistent* if there is no w.f.f. A such that both $\vdash A$ and $\vdash \sim A$. (In what follows a stronger condition called ω-consistency is sometimes required, but we shall forego an explanation of this for the sake of brevity.) A formal system is *complete* if, for every w.f.f. A, either $\vdash A$ or the system would become inconsistent if A were introduced as an additional formal axiom (the reader may think of this as either $\vdash A$ or $\vdash \sim A$, *i.e.*, the system is complete whenever every w.f.f. is theoretically "decidable," although strictly speaking the two concepts do not coincide in all formal systems).

The solution. Our problem now becomes the problem of setting up a formal system which is *adequate* for the derivation of mathematics, and which further is *consistent* (*i.e.*, leads to no contradiction) and *complete* (*i.e.*, we would like every mathematical proposition to be at least theoretically decidable in the sense that it can either be proved or refuted). The problem of adequacy was solved by Frege, Whitehead, and Russell. (Of course, the adequacy of their systems was demonstrated for extant mathematics only. Nothing can be ascertained concerning the adequacy for all future developments of mathematics.) Several systems have been proposed which it is hoped are consistent as well as adequate. To mention a few, there is Russell and Whitehead's *Principia Mathematica*, Quine's *Mathematical Logic*, and systems by Bernays, Fraenkel, Gödel, von Neumann, Zermelo and others. Although it is hoped that these systems are consistent, none have actually been proved to be so.

The consistency problem was partially reduced in classical mathematics by the so-called "method of models." Thus Beltrami in 1868 reduced the consistency of Lobatchevsky's non-Euclidean geometry to that of Euclidean geometry. In this and other manners, the consistency of large parts of classical mathematics is ultimately reducible to that of the arithmetic of natural numbers, particularly as this theory is embodied in the Peano axioms, or in a set theory sufficiently comprehensive for the derivation of the Peano axioms.

For several inadequate partial systems the consistency problem has been

successfully solved. Post in 1921 demonstrated both the consistency and the completeness (in a somewhat different sense than that defined above) of the propositional calculus. The first order functional calculus was proved consistent in 1928 by Hilbert and Ackermann, and complete (in still another sense) in 1930 by Gödel. More recently Tarski has demonstrated the consistency and completeness of the arithmetic of real numbers, and also of elementary geometry.

With this background of success it was widely felt that the solution of the consistency problem for the arithmetic of natural numbers was essentially at hand, that it was now merely a matter of detail. The necessary tools had already been forged in Hilbert's new branch of research, metamathematics, and few doubted that a positive solution to the problem was already within reach. Surprisingly however, despite intensive efforts by Hilbert and others, and very close approaches to a solution, the problem remained refractory.

This puzzling situation was illuminated in 1931 by the appearance of certain remarkable and unexpected results of Gödel which solved in a negative fashion both the consistency and the completeness problem. These theorems, among the most remarkable in all of mathematics, revealed an unforeseen limitation in the methods of formal mathematics in some ways analogous to the limitations on physical measurements imposed by the Heisenberg Uncertainty Principle. Working within systems either presupposing the Peano axioms or possessing a syntax sufficiently rich for the derivation of these axioms—systems related to the *Principia Mathematica*—Gödel showed the impossibility of proving the consistency of such systems by so-called "finitary" methods, that is by constructive methods formalizable within the system itself. So the proof of consistency which seemed so close to attainment is actually impossible of attainment, at least in the form originally envisioned by Hilbert.

This remarkable result is but a consequence of even more fundamental results concerning the completeness of the formal systems studied by Gödel. Gödel demonstrated the incompleteness of such systems, that is, he revealed the existence of "undecidable" propositions, of which consistency is one. This result is particularly surprising in view of the fact that the formal systems with which Gödel worked are sufficiently rich in syntax for the derivation of all of classical mathematics (and presumably of much of mathematics yet to be developed). Yet despite this richness of syntax, it is possible to formulate within these systems arithmetical problems which are theoretically unsolvable within the system.

No attempt will be made here to outline Gödel's demonstrations, for even such an outline would of necessity be long, detailed, and technical. Instead we will content ourselves with a heuristic account of Gödel's reasoning which we hope will at least reveal the "point" of the proof.

Suppose we denote by "*CONSIS*" a syntactical formula whose semantic interpretation implies the consistency of the formal system (for example, *CONSIS* may mean "there is no proof for $1=0$"). The consistency problem would then be related to the problem of demonstrating $\vdash$ *CONSIS*. One pos-

sible line of attack on the problem would be to seek a sufficient condition A such that $\vdash(A \rightarrow CONSIS)$ and then to try to demonstrate $\vdash A$. $\vdash CONSIS$ would then follow by *modus ponens*.

Now Gödel foresaw what others had failed to suspect, namely that *CONSIS* was neither provable nor refutable if the system is truly consistent. To demonstrate this he hit upon the following ingenious scheme. Suppose we seek a condition B, necessary for *CONSIS* [so that $\vdash(CONSIS \rightarrow B)$] and then show that B is true but unprovable within the system. It would then follow that *CONSIS* is undecidable, for if $\vdash CONSIS$, then B would be provable by *modus ponens*, which would be a contradiction.

The question now remains, can we construct such a formal statement, or formula B which, assuming consistency, is true but unprovable within the formal system? If we could, we will have solved in a negative fashion both the consistency problem and the completeness problem, *i.e.*, we will have shown that the methods available in a formal system adequate for extant mathematics are inadequate to demonstrate its own consistency, and thus inadequate to decide even theoretically all mathematical propositions.

Taking a hint from the Epimenides antinomy, Gödel constructed such an undecidable formula B by constructing a formula whose semantical content is that B is undecidable. Consider the Epimenidean statement: "This statement is false." By analogy, we might try "This statement can neither be proved nor disproved" for our statement B. However, one difficulty in following this Epimenidean construction in the construction of our Gödelian statement B is that the self-reference occurring in the Epimenidean statement is semantical in character. Such semantical self-reference is of course precluded in a formal system. We must seek a syntactical device which will enable statement B to refer to itself in a purely structural manner.

A semi-syntactical device is available in the English language through the use of quotation marks. The following statement, for example, is Gödelian in character:

> "yields a statement which can neither be proved nor disproved when appended to its own quotation" yields a statement which can neither be proved nor disproved when appended to its own quotation.

Of course the device of quotation is a part of the syntax of natural languages and not of mathematics, formal or otherwise, so it was necessary to seek further. Gödel accomplished his aim by a very elaborate and ingenious numerical device, assigning coordinates to the decidable formulas of the formal system, establishing an enumeration of these formulas, and constructing the undecidable formula B by an application of Cantor's diagonal process.

These results on incompleteness refer of course only to certain formal systems whose syntax is adequate for the formalization of many properties of the natural numbers used in the construction of the "undecidable" proposition B (of the Chinese remainder theorem, and of the possibility of mapping integers onto finite sequences of integers, for example). It might be thought that the

defect of the incompleteness of such systems could be remedied by the addition of further effective lists of axioms which would enable one to derive the propositions which were previously undecidable. But very surprisingly even this expedient, for all its apparent reasonableness, fails utterly. For Gödel has further shown that not only are the formal systems incomplete but also they are *incompletable* in the sense that new undecidable propositions can always be constructed within the system enlarged by the addition of these new axioms.

A final word concerning the significance of these syntactical results for ordinary intuitive mathematics. Gödel has shown that the formal systems known to be adequate for the derivation of mathematics are unsafe in the sense that their consistency cannot be demonstrated by finitary methods formalizable within the system, whereas any system known to be safe in this sense is inadequate. We are left in the position of the student of elementary mathematics who "knows what a logarithm is, but just can't put it into words." We know, or at least strongly suspect, mathematics to be consistent. But we just cannot incorporate this knowledge into a formal system.

The methods used in Gödel's demonstrations were constructive and finitary in character, and consequently acceptable even to the most cautious of the critics of mathematical methods. In view of Gödel's results, if a proof of consistency is to be forthcoming it must employ methods which are not formalizable within the system. Such proofs have been devised. Thus Gentzen has proved the consistency of number theory using transfinite induction up to a sufficiently large ordinal—a method of proof not formalizable within the system whose consistency it proves. While such methods are perhaps acceptable to most mathematicians, the acceptance is by no means universal. Indeed there are those to whom no proof will be acceptable except one of a type explicity ruled out by Gödel's discoveries. In view of the lack of a universally acceptable proof, belief in the consistency of mathematics becomes then somewhat a matter of faith rather than reason.

Suppose we loosely define a *religion* as any discipline whose foundations rest on an element of faith, irrespective of any element of reason which may be present. Quantum mechanics for example would be a religion under this definition. But mathematics would hold the unique position of being the only branch of theology possessing a rigorous demonstration of the fact that it should be so classified.

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A CHAIN OF CIRCLES ASSOCIATED WITH THE n-LINE

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1. Introduction. The Clifford-de Longchamps-Morley-Bath series of theorems in the geometry of the n-line are well known and were stated in an earlier paper.* We now add another chain of circles to the system. In section 2 an analytic method is set up and applied to the Morley circles. In sections 3 and 4 the de Longchamps points and the Clifford points and circles are discussed. In sections 5 and 6 a chain of circles which we shall call the Miquel circles and a chain of points which we shall call the Hermes points are defined. In section 7 the new chain of circles is reached and its chief properties proved. In the final sections the special cases of the 5-line and the 6-line are considered.

2. The Morley circles.† We take n lines in the plane, no three concurrent and no two parallel, and consider the "normal" equation of l_f: $x \cos \alpha_f + y \sin \alpha_f = p_f$. If $z = x + iy$ and $\bar{z} = x - iy$, this can be put into the form $z + \mathbf{f}^2\bar{z} = k_f\mathbf{f}$, where $\mathbf{f}$ is an abbreviation for $\cos \alpha_f + i \sin \alpha_f$, and k_f is twice p_f. Throughout this paper the boldface numerals $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{n}$ are used in this sense. For example $\mathbf{1}^3 = \cos 3\alpha_1 + i \sin 3\alpha_1$.

Thus l_1 is $z + 1^2\bar{z} = k_1\mathbf{1}$ and l_2 is $z + 2^2\bar{z} = k_2\mathbf{2}$. Then A_{12}, the intersection of l_1 and l_2 is found to be $z = -\mathbf{12}\{k_1\mathbf{2} - k_2\mathbf{1}\}/(\mathbf{12})$, where $(\mathbf{12})$ is an abbreviation for $1^2 - 2^2$. We shall use $(\mathbf{123})$ as an abbreviation for $(\mathbf{12})(\mathbf{13})(\mathbf{23})$, $(\mathbf{2346})$ for $(\mathbf{23})(\mathbf{24})(\mathbf{26})(\mathbf{34})(\mathbf{36})(\mathbf{46})$, and so on. The numerals will always be placed in ascending order.

If, as in this case, z is a function of k's and boldface numerals, $\bar{z}$ can be found by replacing $\mathbf{1}, \mathbf{2}, \cdots, \mathbf{n}$ by $\mathbf{1/1}, \mathbf{1/2}, \cdots, \mathbf{1/n}$, since $\cos \alpha_f - i \sin \alpha_f = 1/(\cos \alpha_f + i \sin \alpha_f)$. Thus, for A_{12}, $\bar{z} = \{k_1\mathbf{1} - k_2\mathbf{2}\}/(\mathbf{12})$.

The square of the distance between two points (x_1, y_1) and (x_2, y_2) is given by $(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$. Hence the equation of the perpendicular bisector of $A_{12}A_{23}$ is

$$\left[z + \mathbf{12}\frac{k_1\mathbf{2} - k_2\mathbf{1}}{(\mathbf{12})}\right]\left[\bar{z} - \frac{k_1\mathbf{1} - k_2\mathbf{2}}{(\mathbf{12})}\right] = \left[z + \mathbf{23}\frac{k_2\mathbf{3} - k_3\mathbf{2}}{(\mathbf{23})}\right]\left[\bar{z} - \frac{k_2\mathbf{2} - k_3\mathbf{3}}{(\mathbf{23})}\right].$$

This reduces to

$$z - 2^2\bar{z} = \frac{-2}{(\mathbf{12})(\mathbf{23})}\left[k_1\mathbf{12}(\mathbf{23}) - k_2\{2^4 - 1^2 3^2\} - k_3\mathbf{23}(\mathbf{12})\right].$$

Similarly the perpendicular bisector of $A_{12}A_{23}$ is

* This MONTHLY, vol. 61, 1954, p. 161.

† These circles were first announced by de Longchamps, Nouv. Corres. de Math., 1877, p. 306. Because of an important paper by F. Morley, Trans. Amer. Math. Soc., 1900, p. 99, they have been called Morley circles. R. Goormaghtigh in a paper in this MONTHLY, vol. 54, 1947, p. 327, calls them the centric circles. Other references are found in H. F. Baker, Introduction to Pure Geom., note 1, p. 330, Cambridge, 1943.

$$z - 3^2\bar{z} = \frac{-3}{(13)(23)}\left[k_1 13(23) + k_2 23(13) + k_3\{3^4 - 1^2 2^2\}\right].$$

Eliminating $\bar{z}$, we find that C_{123}, the center of the circle circumscribing $A_{12}A_{13}A_{23}$ is

$$123\left\{\frac{k_1 23(23) - k_2 13(13) + k_3 12(12)}{(123)}\right\}.$$

The radius of this circle is

$$123\left\{\frac{k_1 1(23) - k_2 2(13) + k_3 3(12)}{(123)}\right\}.$$

The equation of the circle is

$$\begin{aligned}(123)z\bar{z} - z\{k_1 1^3(23) - k_2 2^3(13) + k_3 3^3(12)\}& \\ - \bar{z}123\{k_1 23(23) - k_2 13(13) + k_3 12(12)\}& \\ = 123\{k_1 k_2 3(12) - k_1 k_3 2(13) + k_2 k_3 1(23)\}.&\end{aligned}$$

We shall abbreviate these expressions so that C_{123} is $123\sum k_1 23(23)/(123)$, r_{123} is $123\sum k_1 1(23)/(123)$, and the circle is

$$(123)z\bar{z} - z\sum k_1 1^3(23) - \bar{z}123\sum k_1 23(23) = 123\sum k_1 k_2 3(12).$$

In these summations the numerals in the subscripts of the k's and in the boldface numbers are to be rearranged in every possible way except that they must always be in ascending order among the k's and the boldface numbers respectively; the sign of a term is plus or minus according as the number of transpositions needed to put the numerals back into the original order is even or odd.

We shall use $\mathbf{n}!$ for $\mathbf{12\cdots n}$, $\mathbf{n}!/\mathbf{f}$ for $\mathbf{n}!$ with $\mathbf{f}$ omitted, $\mathbf{n}!/\mathbf{fg}$ for $\mathbf{n}!$ with $\mathbf{f}$ and $\mathbf{g}$ omitted, $\mathbf{n}!/\mathbf{f}!$ for $\overline{\mathbf{f+1}}\ \overline{\mathbf{f+2}}\cdots\mathbf{n}$ and so on. The same abbreviations will be used for integers.

It may be noted that $(\mathbf{n}!/12)(\mathbf{n}!) = (\mathbf{n}!/1)(\mathbf{n}!/2)(12)$. We now set up a chain of points as follows. $C_{n!}$ is defined as $(-1)^{n-1}\mathbf{n}!\sum k_1\mathbf{n}!/1(\mathbf{n}!/1)/(\mathbf{n}!)$, $C_{n!/f}$ as $(-1)^n\mathbf{n}!/\mathbf{f}\sum k_1\mathbf{n}!/1\mathbf{f}(\mathbf{n}!/1\mathbf{f})/(\mathbf{n}!/\mathbf{f})$.

If we take the first of these to be z_1 and the second z_2, then

$$\bar{z}_1 = \frac{\sum k_1 1^{2n-3}(\mathbf{n}!/1)}{(\mathbf{n}!)} \quad \text{and} \quad \bar{z}_2 = \frac{\sum k_1 1^{2n-3}(\mathbf{n}!/1\mathbf{f})}{(\mathbf{n}!/\mathbf{f})}.$$

Then $z_1 - z_2 = (-1)^{n-1}\{\mathbf{n}!/\mathbf{f}\}^2\sum k_1 1(\mathbf{n}!/1)/(\mathbf{n}!)$, $\bar{z}_1 - \bar{z}_2 = \mathbf{f}^2\sum k_1 1^{2n-5}(\mathbf{n}!/1)/(\mathbf{n}!)$. Then the square of the distance between $C_{n!}$ and $C_{n!/f}$ is $(-1)^{n-1}\{\mathbf{n}!\}^2/(\mathbf{n}!)^2 \cdot \sum k_1 1(\mathbf{n}!/1)\cdot\sum k_1 1^{2n-5}(\mathbf{n}!/1)$.

Since this is independent of $\mathbf{f}$, it follows that $C_{n!}$ is the center of a circle passing through the points $C_{n!/1}, C_{n!/2}, \cdots, C_{n!/n}$. It follows that $\mathfrak{C}_{1234}$ passes through C_{123}, C_{124}, C_{134}, C_{234} and is therefore the Morley circle of the 4-line; $\mathfrak{C}_{12345}$ passes

through C_{1234}, C_{1235}, C_{1245}, C_{1345}, C_{2345} and is therefore the Morley circle of the 5-line; and so on indefinitely.

We also note that the square of the radius of the circle $\mathfrak{C}_{n!/f}$ is

$$(-1)^n \frac{\{n!/f\}^2}{(n!/f)^2} \sum k_1 1(n!/1f) \cdot \sum k_1 1^{2n-7}(n!/1f).$$

3. The de Longchamps points.* Consider the circles $\mathfrak{C}_{123}$ and $\mathfrak{C}_{124}$:

$$(123)z\bar{z} - z \sum k_1 1^3(23) - \bar{z}123 \sum k_1 23(23) = 123 \sum k_1 k_2 3(12)$$
$$(124)z\bar{z} - z \sum k_1 1^3(24) - \bar{z}124 \sum k_1 24(24) = 124 \sum k_1 k_2 4(12).$$

Eliminating $\bar{z}$, we have

$$\frac{(123)z - 123 \sum k_1 23(23)}{(124)z - 124 \sum k_1 24(24)} = \frac{z \sum k_1 1^3(23) + 123 \sum k_1 k_2 3(12)}{z \sum k_1 1^3(24) + 124 \sum k_1 k_2 4(12)}.$$

This reduces to

$$[z(12) + 12\{k_1 2 - k_2 1\}][z \sum k_1 1^3(234) - 1234 \sum k_1 k_2 34(12)(34)] = 0.$$

This means that these circles intersect at A_{12} and also at a point which we shall call K_{1234} which is given by

$$\frac{1234 \sum k_1 k_2 34(12)(34)}{\sum k_1 1^3(234)}.$$

Evidently $\mathfrak{C}_{134}$ and $\mathfrak{C}_{234}$ also pass through this point. This is the de Longchamps point of the 4-line.

We now set up a chain of points $K_{n!}$, defined as

$$\frac{(-1)^n n! \sum k_1 k_2 n!/12(n!/12)\{1^{2n-6} - 2^{2n-6}\}}{\sum k_1 1^{2n-5}(n!/1)}.$$

If we use for z_1 and $\bar{z}_1$ the values given in the last section for $C_{n!,f}$ and for z_2 the value given above for $K_{n!}$, then

$$\bar{z}_2 = -\frac{\sum k_1 k_2 12(n!/12)\{1^{2n-6} - 2^{2n-6}\}}{\sum k_1 1(n!/1)}.$$

Then

$$z_1 - z_2 = \frac{(-1)^n\{n!/f\}^2 \cdot \sum k_1 1(n!/1) \cdot \sum k_1 1^{2n-7}(n!/1f)}{(n!/f) \sum k_1 1^{2n-5}(n!/1)};$$

$$\bar{z}_1 - \bar{z}_2 = \frac{\sum k_1 1^{2n-5}(n!/1) \cdot \sum k_1 1(n!/1f)}{(n!/f) \sum k_1 1(n!/1)}.$$

Hence the square of the distance between these two points is

* de Longchamps, *loc. cit.* Goormaghtigh calls these points the centric foci of the n-line.

$$\frac{(-1)^n\{n!/f\}^2}{(n!/f)^2}\cdot\sum k_1 1(n!/1f)\cdot\sum k_1 1^{2n-7}(n!/1f).$$

Comparing this with the result given at the end of section 2, we see that $K_{n!}$ lies on the Morley circles $\mathfrak{C}_{n!,1}$, $\mathfrak{C}_{n!,2}$, $\cdots$, $\mathfrak{C}_{n!,n}$. Thus $\mathfrak{C}_{1234}$, $\mathfrak{C}_{1235}$, $\mathfrak{C}_{1245}$, $\mathfrak{C}_{1345}$, $\mathfrak{C}_{2345}$, all pass through K_{12345}; and so on indefinitely.

4. Clifford points and circles.† We shall not be concerned greatly with these. However, if we take the point F_{12345} to be

$$\frac{-12345\sum k_1k_2345(12)(345)}{\sum k_1 1^3(2345)},$$

we may readily prove by the methods used above that F_{12345} is the center of a circle passing through K_{1234}, K_{1235}, K_{1245}, K_{1345}, K_{2345}. This is the Clifford circle of the 5-line. The square of its radius is

$$\frac{-\{12345\}^2[\sum k_1k_212(12)(345)]^2}{[k_1 1^3(2345)]^2}.$$

It may readily be verified that this circle also passes through K_{12345}, which is Bath's theorem.‡

Similarly it may be shown that F_{123456}, which is given by

$$\frac{-123456\sum k_1k_2k_3456(123)(456)}{\sum k_1k_2 1^3 2^3(12)(3456)},$$

lies on this circle and also on the Clifford circles $\mathfrak{F}_{12346}$, $\mathfrak{F}_{12356}$, $\mathfrak{F}_{13456}$, $\mathfrak{F}_{12456}$, $\mathfrak{F}_{23456}$.

This can be generalized to prove the existence of the Clifford circle of the $\overline{2p+1}$-line and the Clifford point of the $\overline{2p}$-line.

5. Miquel circles. We call the circle concentric with the Morley circle of the n-line and passing through its de Longchamps point, the Miquel circle§ of the n-line. Taking $C_{n!}$ to be z_1 and $K_{n!}$ to be z_2, we find that

$$z_1 - z_2 = \frac{(-1)^{n-1}\{n!\}^2\cdot\sum k_1 1^{2n-7}(n!/1)\cdot\sum k_1 1(n!/1)}{(n!)\sum k_1 1^{2n-5}(n!/1)}$$

$$\bar{z}_1 - \bar{z}_2 = \frac{\sum k_1 1^3(n!/1)\cdot\sum k_1 1^{2n-5}(n!/1)}{(n!)\sum k_1 1(n!/1)}.$$

Hence the square of the radius of the Miquel circle is

$$\frac{(-1)^{n-1}\{n!\}^2\sum k_1 1^3(n!/1)\cdot\sum k_1 1^{2n-7}(n!/1)}{(n!)^2}.$$

† Clifford, Mess. Math., 1871.

‡ Bath, Proc. Camb. Phil. Soc., 1939, p. 518.

§ Miquel, Jour. de Math., 1838, p. 485, announced important properties of the 4- and 5-lines. He did not make explicit reference to these circles.

Note that when $n=4$, the Miquel circle coincides with the Morley circle.

6. Hermes points. Consider the circles $\mathfrak{C}_{1234}$ and $\mathfrak{C}_{123}$.

$$(1234)z\bar{z}-z\sum k_1 1^5(234)+\bar{z}1234\sum k_1 234(234)=-1234\sum k_1k_2 34(34)\{1^4-2^4\}$$
$$(123)z\bar{z}-z\sum k_1 1^3(23)-\bar{z}123\sum k_1 23(23)=123\sum k_1k_2 3\{1^2-2^2\}.$$

Eliminating $\bar{z}$,

$$\frac{(1234)z+1234\sum k_1 234(234)}{(123)z-123\sum k_1 23(23)}=\frac{z\sum k_1 1^5(234)-1234\sum k_1k_2 34(34)\{1^4-2^4\}}{z\sum k_1 1^3(23)+123\sum k_1k_2 3\{1^2-2^2\}}.$$

This reduces to

$$[z\sum k_1 1^3(234)-1234\sum k_1k_2 34(34)(12)]$$
$$[z4^2(123)-123\sum k_1 23(23)\{1^2+4^2\}]=0.$$

The Miquel circle of the 4-line thus intersects the Morley circle of the 3-line obtained by omitting l_4 in the de Longchamps point of the 4-line and also in the point given by

$$\frac{123\sum k_1 23(23)\{1^2+4^2\}}{4^2(123)}.$$

We shall call this one of the Hermes points of the 4-line and denote it by $H_{4:123}$. There are four Hermes points of the 4-line, $H_{4:123}$, $H_{3:124}$, $H_{2:134}$, $H_{1:234}$.*

We shall now consider the point $H_{f:\,n!/f}$ which we shall take to be

$$\frac{(-1)^n n!/f\sum k_1 n!/1f(n!/1f)\{1^2+f^2\}}{f^2(n!/f)}.$$

We wish to show that this represents a Hermes point of the n-line, which we define as the intersection other than $K_{n!}$ of the Morley circle of the $\overline{n-1}$-line $\mathfrak{C}_{n!/f}$, and the Miquel circle of the n-line.

We take z_1 to be the value just given. Then

$$\bar{z}_1=\sum k_1 1^{2n-7}(n!/1f)\{1^2+f^2\}/(n!/f).$$

For z_2 and $\bar{z}_2$ we take the values for $C_{n!/f}$ given in section 2. Then

$$z_1-z_2=\frac{(-1)^n\{n!/f\}^2\sum k_1 1(n!/1f)}{f^2(n!/f)};\quad \bar{z}_1-\bar{z}_2=\frac{f^2\sum k_1 1^{2n-7}(n!/1f)}{(n!/f)}.$$

Then the square of the distance between $H_{f:\,n!/f}$ and $C_{n!/f}$ is

$$\frac{(-1)^n\{n!/f\}^2\sum k_1 1(n!/1f)\cdot\sum k_1 1^{2n-7}(n!/1f)}{(n!/f)^2}.$$

* Hermes, Nouv. Ann., 1859, p. 359.

But this is the square of the radius of $\mathfrak{C}_{n!/f}$. Hence $H_{f:n!/f}$ lies on this circle.

Again, let z_1 and $\bar{z}_1$ have the same values as above, but give z_2 and $\bar{z}_2$ the values given in section 2 for $C_{n!}$. Now

$$z_1 - z_2 = \frac{(-1)^n\{n!/f\}^2 \sum k_1 1^3(n!/1)}{f^2(n!)}; \qquad \bar{z}_1 - \bar{z}_2 = \frac{-f^4 \sum 1^{2n-7}(n!/1)}{(n!)}.$$

Then the square of the distance between $H_{f:n!/f}$ and $C_{n!}$ is

$$\frac{(-1)^{n-1}\{n!\}^2 \sum k_1 1^3(n!/1) \cdot \sum k_1 1^{2n-7}(n!/1)}{(n!)^2}.$$

But this is the value obtained in section 5 for the square of the radius of the Miquel circle with center $C_{n!}$. Hence $H_{f:n!/f}$ gives the n Hermes points of the n-line by taking $f = 1, 2, \cdots, n$.

7. The new chain of circles. We are now ready to set up the new chain of circles which it is the main purpose of this paper to announce.

These circles have $H_{f:n!/f}$ for center and pass through $K_{n!/f}$.

We must first find the square of the radius of this circle. Here z_1 and $\bar{z}_1$ have the values given in the last section for $H_{f:n!/f}$ and

$$z_2 = \frac{(-1)^{n-1}n!/f \sum k_1 k_2 n!/12f(n!/12f)\{1^{2n-8} - 2^{2n-8}\}}{\sum k_1 1^{2n-7}(n!/1f)}$$

$$\bar{z}_2 = \frac{-\sum k_1 k_2 12(n!/12f)\{1^{2n-8} - 2^{2n-8}\}}{\sum k_1 1(n!/1f)}.$$

Then

$$z_1 - z_2 = \frac{(-1)^n\{n!/f\}^2 \sum k_1 1(n!/1f) \cdot \sum k_1 1^{2n-9}(n!/1f)\{1^2 + f^2\}}{f^2(n!/f) \sum k_1 1^{2n-7}(n!/1f)};$$

and

$$\bar{z}_1 - \bar{z}_2 = \frac{\sum k_1 1^{2n-7}(n!/1f) \cdot \sum k_1 1(n!/1f)\{1^2 + f^2\}}{(n!/f) \sum k_1 1(n!/1f)}.$$

Then the square of the radius is

$$\frac{(-1)^n\{n!/f\}^2 \sum k_1 1(n!/1f)\{1^2 + f^2\} \cdot \sum k_1 1^{2n-9}(n!/1f)\{(1^2 + f^2)\}}{f^2(n!/f)^2}.$$

We shall now show that this circle also passes through the points $H_{f:n!/fg}$, where g is any value from 1 to n other than f.

Here z_1 and $\bar{z}_1$ have the values used above, while

$$z_2 = \frac{(-1)^{n-1}n!/fg \sum k_1 n!/1fg(n!/1fg)\{1^2 + f^2\}}{f^2(n!/fg)}$$

$$\bar{z}_1 = \frac{\sum k_1 1^{2n-9}(n!/1fg)\{1^2 + f^2\}}{(n!/fg)}.$$

$$z_1 - z_2 = \frac{(-1)^n\{n!/fg\}^2 \sum k_1 1(n!/1f)\{1^2 + f^2\}}{f^2(n!/f)};$$

$$\bar{z}_1 - \bar{z}_2 = \frac{g^2 \sum k_1 1^{2n-9}(n!/1f)\{1^2 + f^2\}}{(n!/f)}.$$

And the square of the distance between $H_{f:\,n!/f}$ and $H_{f:\,n!/fg}$ turns out to be the same as that given above. Hence we have proved

THEOREM 1. *Each of the n Hermes points of an n-line, such as* $H_{f:\,n!/f}$, *f being any integer from* 1 *to n (where the Miquel circle whose center is* $C_{n!}$ *is intersected by the Morley circle* $\mathfrak{C}_{n!/f}$), *is the center of a circle which passes through the de Longchamps point* $K_{n!/f}$ *and also through* $n-1$ *Hermes points* $H_{f:\,n!/1f}$, $H_{f:\,n!/2f}$, $\cdots$, $H_{f:\,n!/fn}$.

Using these methods we may show that the equation of the circle with $H_{f:\,n!/f}$ as center and passing through $K_{n!/f}$ is

$$(n!/f)f^2z\bar{z} - zf^2 \sum k_1 1^{2n-7}(n!/1f)\{1^2+f^2\} - \bar{z}(-1)^n n!/f \sum k_1 n!/1f(n!/1f)\{1^2+f^2\}$$
$$= (-1)^n n!/f \sum k_1 k_2 n!/12f(n!/12f)\{1^{2n-8} - 2^{2n-8}\}\{1^2+f^2\}\{2^2+f^2\}.$$

In the same way, the circle with $H_{g:\,n!/g}$ as center passing through $K_{n!/g}$ may be written down by replacing **f** by **g** in the above.

Multiplying the first of these equations by $g^2(1f)(2f)\cdots(fn)$ and the second by $f^2(1g)(2g)\cdots(gn)$ and subtracting, we obtain the equation of the radical axis of these two circles to be

$$zf^2g^2 \sum k_1 1^{2n-7}(n!/1) - \bar{z}(-1)^n\{n!/fg\}^2 \sum k_1 1^3(n!/1)$$
$$= (-1)^{n-1} n!/fg \sum k_1 k_2 n!/12fg(n!/12)\{1^{2n-8} - 2^{2n-8}\}\{1^4 2^4 - f^4 g^4\}.$$

Similarly, the radical axis of the circles of this system with centers $H_{f:\,n!/f}$ and $H_{h:\,n!/h}$ can be written down by substituting **h** for **g** in the above expression.

Eliminating $\bar{z}$ between these two equations, we find that

$$z = \frac{(-1)^n n! \sum k_1 k_2 n!/12(n!/12)\{1^{2n-8} - 2^{2n-8}\}}{\sum k_1 1^{2n-7}(n!/1)}.$$

Since this is independent of **f**, **g** and **h**, it follows that

THEOREM 2. *The n circles of Theorem* 1 *form a system which has a common radical center.*

8. The 5-line. In the special case $n = 5$, the radical center becomes

$$\frac{-12345\sum k_1k_2345(345)\{1^2-2^2\}}{\sum k_11^2(2345)}.$$

But this is the value of F_{12345} given in section 4.

THEOREM 3. *The radical center of the five new circles of the 5-line is the center of the Clifford circle of the 5-line.*

Moreover it may be proved that

THEOREM 4. *The Clifford circle of the 5-line is orthogonal to each of the five new circles of the 5-line.*

It may also be proved laboriously that:

THEOREM 5. *The inverse points of the de Longchamps point of the 5-line with respect to each of its five new circles all lie on the Clifford circle of the 5-line.*

The equation of the straight line $A_{12}C_{124}$ is easily shown to be

$$4^2z - 1^2z^2z = -\frac{12[k_12\{1^2+4^2\} - k_21\{2^2+4^2\}]}{(12)}.$$

This may be proved to pass through $H_{4:123}$. Similarly $A_{13}C_{134}$ and $A_{23}C_{234}$ pass through $H_{4:123}$. This theorem of Hermes is the reason for giving the name of Hermes points to this set of points.

COROLLARY. *$A_{12}C_{123}$ passes through $H_{3:124}$ and $H_{5:125}$.*
$A_{12}C_{124}$ passes through $H_{4:123}$ and $H_{4:125}$.
$A_{12}C_{125}$ passes through $H_{5:123}$ and $H_{5:124}$.

There are three such lines passing through each of the ten vertices of the 5-line.

Again the equation of the straight line $H_{4:123}H_{5:123}$ is

$$4^2z^2z + 1^2z^2z^2z - \frac{123\sum k_123(23)\{1^2+4^2\}\{1^2+5^2\}}{(123)};$$

similarly the equation of the straight line $H_{3:124}H_{5:124}$ is

$$3^2z^2z + 1^2z^2z^2z = \frac{124\sum k_124(24)\{1^2+3^2\}\{1^2+5^2\}}{(124)}.$$

These two lines intersect at $-1234\sum k_1\ 234(234)\{1^2+5^2\}/(1234)$, which is $H_{5:1234}$.

It may be shown that $H_{5:134}H_{2:134}$ and $H_{5:234}H_{1:234}$ also pass through $H_{5:1234}$. This may be generalized as follows.

THEOREM 6. *Through $H_{f:5lll}$ pass four straight lines joining $H_{f:5lll_g}$ and $H_{g:5lll_g}$, where f has any of the values 1, 2, 3, 4 or 5, and g any of these values other than that taken for f.*

COROLLARY. *$H_{f:\,5^1/_{f\theta}}$ and $H_{\sigma:\,5^1/_{f\theta}}$ lie on the line of centers of the new circles with centers $H_{f:\,5^1/_f}$ and $H_{\sigma:\,5^1/_\theta}$.*

9. The 6-line. In the special case $n=6$, the radical center becomes

$$\frac{123456\sum k_1k_23456(3456)\{1^4-2^4\}}{\sum k_11^5(23456)}.$$

It is easily proved by the methods used above that this point lies on each of the six new circles of the 6-line and also on the Morley circle of the 6-line.

THEOREM 7. *The six new circles of the 6-line intersect at a common point, which also lies on the Morley circle of the 6-line.*

Again, $H_{1:\,2345}H_{2:\,1345}$ is

$$1^22^2z+3^24^25^2\bar{z}=\frac{345\sum k_345(45)\{1^2+3^2\}\{2^2+3^2\}}{(345)};$$

and $H_{1:\,2346}H_{2:\,1346}$ is

$$1^22^2z+3^24^36^2\bar{z}=\frac{346\sum k_346(46)\{1^2+3^2\}\{2^2+3^2\}}{(346)}.$$

These intersect at

$$\frac{-3456\sum k_3456(456)\{1^2+3^2\}\{2^2+3^2\}}{1^22^2(3456)}.$$

Evidently $H_{1:\,2356}H_{2:\,1356}$ and $H_{1:\,2456}H_{2:\,1456}$ also pass through this point. But this point may also be proved to lie on the new circles with $H_{1:\,23456}$ and $H_{2:\,13456}$ as centers. Hence

THEOREM 8. *In a 6-line the two new circles with centers $H_{1:\,23456}$ and $H_{2:\,13456}$ meet in a point other than that common to all six circles and through this point pass also the four lines $H_{1:\,2345}H_{2:\,1345}$, $H_{1:\,2346}H_{2:\,1346}$, $H_{1:\,2356}H_{2:\,1356}$, $H_{1:\,2456}H_{2:\,1456}$.*

This may also be generalized.

It was proved in my previous article* that $A_{12}K_{1245}$, $A_{13}K_{1345}$, $A_{23}K_{2345}$ intersect at a point which we shall call $X_{123,45}$, which is on the Clifford circle of the 5-line. There are ten such points on this circle.

Similarly, $A_{12}K_{1246}$, $A_{13}K_{1346}$, $A_{23}K_{2346}$ intersect at $X_{123,46}$; and $A_{12}K_{1256}$, $A_{13}K_{1356}$, $A_{23}K_{2356}$ intersect at $X_{123,56}$.

Now in the Clifford circle with center F_{12345} which passes through F, the Clifford point of the 6-line,

$$\angle K_{1234}FX_{123,45}=\angle K_{1234}K_{1245}X_{123,45}=\angle K_{1234}K_{1245}A_{12}.$$

And in the Clifford circle with center F_{12346},

* *Loc. cit.*

$$\angle K_{1234}FX_{123,46} = \angle K_{1234}K_{1246}X_{123,46} = \angle K_{1234}K_{1246}A_{12}.$$

But A_{12}, K_{1234}, K_{1245}, K_{1246} all lie on the circle $\mathfrak{C}_{124}$. Hence $\angle K_{1234}FX_{123,45} = \angle K_{1234}FX_{123,46}$. Therefore $X_{123,45}X_{123,46}$ passes through F. Similarly $X_{123,56}$ may be proved to lie on this line. We thus obtain

THEOREM 9. *The sixty points on the* 6 *Clifford circles of the* 5*-lines obtained by omitting in turn each of the sides of the* 6*-line, lie three by three on twenty straight lines all of which pass through the Clifford point of the* 6*-line.*

The analytic proof of this theorem is very laborious. Lachlan, in the preface to his *Modern Pure Geometry** said: "Although Analysis may be more powerful as an instrument of research, . . . it might well be taken as an axiom, based upon experience, that every geometrical theorem admits of a simple and direct proof—by the principles of Pure Geometry." If the writer had possessed enough acumen to develop his other theorems by such methods, a good deal of space might have been saved.

* London, 1893.

THE SOLUTION OF A SPECIAL HEAT AND DIFFUSION EQUATION

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1. Introduction. The solution of the problem discussed here has been used to obtain results of some importance to geophysics [1–3]. Although the derivation has little mathematical novelty, it has never been published, and may prove interesting, if only as a slightly out-of-the-ordinary example for differential equations courses.

The equation to be considered arises in connection with the problem of finding temperatures and heat flows due to radioactivity within a sphere. It is shown in [2] that, assuming spherical symmetry, constancy of the thermal conductivity, *etc.*, the required solution may be obtained by the superposition of a number of solutions of the same general type, namely, solutions of the equations

$$\rho c \frac{\partial \theta}{\partial t} = K\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r}\right) + \begin{cases} 0 & , \quad 0 \leqq r \leqq r_1, \\ H_0 e^{-\lambda t}, & r_1 \leqq r \leqq a, \end{cases} \tag{1}$$

with boundary conditions

$$\begin{aligned} &\theta(a, t) = 0, \\ &\theta(0, t) < \infty, \\ &\theta(r, 0) = 0. \end{aligned} \tag{2}$$

Call the temperatures in the regions $0 \leqq r \leqq r_1$ and $r_1 \leqq r \leqq a$ θ_1 and θ_2 respectively, and let

$$u_1 = r\theta_1, \tag{3}$$

$$u_2 = r\theta_2. \tag{4}$$

Write k for $K/\rho c$ and J for $H_0/\rho c$. Then the problem is to solve the equations

$$\frac{\partial u_1}{\partial t} = k\,\frac{\partial^2 u_1}{\partial r^2}, \tag{5}$$

$$\frac{\partial u_2}{\partial t} = k\,\frac{\partial^2 u_2}{\partial r^2} + Jre^{-\lambda t} \tag{6}$$

subject to the conditions

$$u_1(0, t) = 0, \qquad u_1(r, 0) = 0,$$
$$u_1(r_1, t) = u_2(r_1, t),$$
$$\frac{\partial}{\partial r}u_1(r_1, t) = \frac{\partial}{\partial r}u_2(r_1, t),$$
$$u_2(a, t) = 0, \qquad u_2(r, 0) = 0.$$

The most obvious method of solution is to apply the Laplace Transform, and this will be given here. A second method, which gives the final solution in a slightly neater form, will be discussed a little later.

2. Solution using the Laplace transform.* The subsidiary equations obtained by applying the Laplace transformation to (5), (6), and (7) are

$$p\bar{u}_1 = k\,\frac{d^2\bar{u}_1}{dr^2}, \tag{8}$$

$$p\bar{u}_2 = k\,\frac{d^2\bar{u}_2}{dr^2} + \frac{Jr}{p+\lambda}, \tag{9}$$

where

$$\bar{u}_1(0) = 0, \qquad \bar{u}_2(a) = 0,$$
$$\bar{u}_1(r_1) = \bar{u}_2(r_1), \tag{10}$$
$$\frac{d}{dr}\bar{u}_1(r_1) = \frac{d}{dr}\bar{u}_2(r_1).$$

The general solution of (8) is

$$\bar{u}_1 = c_1e^{qr} + c_2e^{-qr}, \tag{11}$$

* The notation used is the same as that in Carslaw and Jaeger: Conduction of Heat in Solids, Oxford, 1947.

where q stands for $\sqrt{p/k}$. The boundary condition $\bar{u}_1(0)=0$ will be satisfied if this is taken in the form

$$\bar{u}_1 = A \sinh qr. \tag{12}$$

Similarly, the general solution of (9) is

$$\bar{u}_2 = B \sinh qr + C \cosh qr + \frac{rJ}{p(p+\lambda)}. \tag{13}$$

In order for the remaining boundary conditions (10) to be satisfied, the following equations must hold:

$$B \sinh qa + C \cosh qa + \frac{aJ}{p(p+\lambda)} = 0, \tag{14}$$

$$B \sinh qr_1 + C \cosh qr_1 + \frac{r_1 J}{p(p+\lambda)} = A \sinh qr_1, \tag{15}$$

$$Bq \cosh qr_1 + Cq \sinh qr_1 + \frac{J}{p(p+\lambda)} = Aq \cosh qr_1. \tag{16}$$

Hence,

$$\begin{aligned} A &= -\frac{J}{p(p+\lambda)\sinh qa}\left(a - r_1 \cosh q(a-r_1) - \frac{\sinh q(a-r_1)}{q}\right), \\ B &= -\frac{J}{p(p+\lambda)\sinh qa}\left(a - r_1 \cosh qa \cosh qr_1 + \frac{\cosh qa \sinh qr_1}{q}\right) \\ C &= \frac{J}{p(p+\lambda)}\left(\frac{\sinh qr_1}{q} - r_1 \cosh qr_1\right). \end{aligned} \tag{17}$$

Thus the solutions of (8) and (9) satisfying (10) are

$$\bar{u}_1 = \frac{J \sinh qr}{p(p+\lambda)\sinh qa}\left(-a + r_1 \cosh q(a-r_1) + \frac{\sinh q(a-r_1)}{q}\right), \tag{18}$$

$$\begin{aligned} \bar{u}_2 = \frac{J}{p(p+\lambda)}\Bigg[r - \frac{\sinh qr}{\sinh qa}\left(a - r_1 \cosh qa \cosh qr_1 + \frac{\sinh qr_1 \cosh qa}{q}\right) \\ + \cosh qr\left(\frac{\sinh qr_1}{q} - r_1 \cosh qr_1\right)\Bigg]. \end{aligned} \tag{19}$$

The solutions for u_1 and u_2 may now be obtained by inverting the above solutions, using the Fourier-Mellin theorem. The calculation of residues involved is tedious, but elementary, and need not be given here. The poles of the functions are at the points $p=0$, $p=-\lambda$, and $p=-m^2\pi^2k/a^2$, for $m=0, 1, 2, \cdots$.

Thus the solution obtained is in each case in the form of a single first term plus an infinite series, *viz.*:

$$\frac{u_1}{J} = \frac{a\epsilon^{-\lambda t}}{\lambda \sin \eta a}\left[\sin \eta r\left(1 - \frac{r_1}{a}\cos \eta(a - r_1) - \frac{1}{\eta a}\sin \eta(a - r_1)\right)\right] \tag{20}$$

$$- \frac{2a^3}{\pi k}\sum_{m=1}^{\infty} \epsilon^{-(m^2\pi^2 k/a^2)t} \cdot \sin \frac{m\pi r}{a} \cdot \left[\frac{(-1)^m - \frac{r_1}{a}\cos \frac{m\pi r_1}{a} + \frac{1}{m\pi}\sin \frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)}\right],$$

$$\frac{u_2}{J} = \frac{a\epsilon^{-\lambda t}}{\lambda \sin \eta a}\left[\sin \eta r - \frac{r}{a}\sin \eta a + \left(\frac{r_1}{a}\cos \eta r_1 - \frac{1}{\eta a}\sin \eta r_1\right)\sin \eta(a - r)\right] \tag{21}$$

$$- \frac{2a^3}{\pi k}\sum_{m=1}^{\infty} \epsilon^{-(m^2\pi^2 k/a^2)t} \cdot \sin \frac{m\pi r}{a} \cdot \left[\frac{(-1)^m - \frac{r_1}{a}\cos \frac{m\pi r_1}{a} + \frac{1}{m\pi}\sin \frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)}\right]$$

where η stands for $\sqrt{\lambda/k}$.

The solution for θ is therefore*

$$\frac{\theta}{J} = \frac{a\epsilon^{-\lambda t}}{r\lambda \sin \eta a} \cdot \begin{cases} \sin \eta r\left(1 - \frac{r_1}{a}\cos \eta(a - r_1) - \frac{1}{\eta a}\sin \eta(a - r_1)\right) \\ \qquad\qquad \text{for } 0 \leqq r \leqq r_1 \\ \sin \eta r - \frac{r}{a}\sin \eta a + \left(\frac{r_1}{a}\cos \eta r_1 - \frac{1}{\eta a}\sin \eta r_1\right)\sin \eta(a - r) \\ \qquad\qquad \text{for } r_1 \leqq r \leqq a \end{cases} \tag{22}$$

$$- \frac{2a^3}{\pi kr}\sum_{m=1}^{\infty} \epsilon^{-(m^2\pi^2 k/a^2)t} \cdot \sin \frac{m\pi r}{a} \cdot \left[\frac{(-1)^m - \frac{r_1}{a}\cos \frac{m\pi r_1}{a} + \frac{1}{m\pi}\sin \frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)}\right].$$

3. Second method of solution. The second method of solution† is simply to try a solution of equations (5) and (6) of the form

$$u_1' = R_1(r)\epsilon^{-\lambda t}, \tag{23}$$

$$u_2' = R_2(r)\epsilon^{-\lambda t}. \tag{24}$$

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* This solution was first obtained by G. Comenetz [1]. That for the simpler problem with $r_1 = 0$ has been worked out independently by H. C. Urey, and is given in his book, The Planets, p. 52. Both can be obtained quite easily from a very general solution by A. N. Lowan, Phys. Rev., vol. 44, 1933, p. 769.

† This was kindly suggested to me by Professor G. F. Duff.

Then R_1 and R_2 satisfy

$$\frac{d^2R_1}{dr^2} + \eta^2 R_1 = 0, \tag{25}$$

$$\frac{d^2R_2}{dr^2} + \eta^2 R_2 + Jr = 0, \tag{26}$$

where

$$\begin{gathered} R_1(0) = 0, \qquad R_1(a) = 0, \\ R_1(r_1) = R_2(r_1), \\ \frac{dR_1(r_1)}{dr} = \frac{dR_2(r_1)}{dr}. \end{gathered} \tag{27}$$

These equations can be solved in much the same manner as equations (8) and (9), giving

$$R_1 = \frac{Ja \sin \eta r}{\lambda \sin \eta a}\left(1 - \frac{r_1}{a}\cos \eta(a - r_1) - \frac{1}{\eta a}\sin \eta(a - r_1)\right) \tag{28}$$

$$R_2 = \frac{Ja}{\lambda \sin \eta a}\left(\sin \eta r - \frac{r}{a}\sin \eta a + \left(\frac{r_1}{a}\cos \eta r_1 - \frac{1}{\eta a}\sin \eta r_1\right)\sin \eta(a - r)\right). \tag{29}$$

The corresponding solutions u_1' and u_2' satisfy (5) and (6) and all of the boundary conditions (7) save the two

$$u_1(r, 0) = 0, \qquad u_2(r, 0) = 0.$$

However, this is easily corrected, for if

$$u = u' + U, \tag{30}$$

where

$$u' = \begin{cases} u_1' & 0 \leqq r \leqq r_1 \\ u_2' & r_1 \leqq r \leqq a, \end{cases} \tag{31}$$

then in order for u to satisfy (5), (6), and (7) completely, U must satisfy

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial r^2}, \tag{32}$$

where

$$\begin{gathered} U(a, t) = 0, \qquad U(0, t) = 0, \\ U(r, 0) = -u'(r, 0). \end{gathered} \tag{33}$$

This of course is a standard heat conduction problem, and the solution is

$$U = \sum_{m=1}^{\infty} A_m e^{-(m^2\pi^2 k/a^2)t} \sin \frac{m\pi r}{a}, \tag{34}$$

where

$$A_m = \frac{2}{a} \int_0^a U(r, 0) \sin \frac{m\pi r}{a}\, dr. \tag{35}$$

The calculation of the A_m is fairly easy, and the final solution obtained is the same as (22), where now the summation term corresponds to U and the first term to u'.

However, this method of solution points to a neater and better way of writing (22), for from (33)

$$u'(r, 0) = -U(r, 0).$$

But

$$\begin{aligned} u'(r, t) &= u'(r, 0)e^{-\lambda t} \\ &= -U(r, 0)e^{-\lambda t} \\ &= \frac{2a^3}{\pi k r} \sum_{m=1}^{\infty} e^{-\lambda t} \cdot \sin \frac{m\pi r}{a} \cdot \left[\frac{(-1)^m - \frac{r_1}{a} \cos \frac{m\pi r}{a} + \frac{1}{m\pi} \sin \frac{m\pi r_1}{a}}{m(a^2\eta^2 - m^2\pi^2)} \right]. \end{aligned} \tag{36}$$

Hence the final solution can be written

$$\begin{aligned} \frac{\theta}{J} = \frac{2a^3}{\pi r} \sum_{m=1}^{\infty} &(e^{-\lambda t} - e^{-(m^2\pi^2 k/a^2)t} \cdot \sin \frac{m\pi r}{a} \\ &\cdot \left[\frac{(-1)^m - \frac{r_1}{a} \cos \frac{m\pi r_1}{a} + \frac{1}{m\pi} \sin \frac{m\pi r_1}{a}}{m(a^2\lambda - m^2\pi^2 k)} \right] \end{aligned} \tag{37}$$

(remembering $\eta^2 = \lambda/k$).

4. Remarks. The heat flow can at once be found from (37), *viz.*

$$\begin{aligned} F(r, t) &= -K \frac{\partial \theta}{\partial r} \\ &= \frac{2a^3 H_0 k}{\pi r} \sum_{m=1}^{\infty} (e^{-\lambda t} - e^{-(m^2\pi^2 k/a^2)t}) \cdot \left(\frac{1}{r} \sin \frac{m\pi r}{a} - \frac{m\pi}{a} \cos \frac{m\pi r}{a} \right) \\ &\quad \cdot \left[\frac{(-1)^m - \frac{r_1}{a} \cos \frac{m\pi r_1}{a} + \frac{1}{m\pi} \sin \frac{m\pi r_1}{a}}{m(a^2\lambda - m^2\pi^2 k)} \right]. \end{aligned} \tag{38}$$

The average temperature is also of interest. This is given by

$$\bar{\theta} = \frac{3}{4\pi a^3}\int_0^a 4\pi r^2\theta\,dr$$

$$= \frac{6a^2J}{\pi^2}\sum_{m=1}^{\infty}(e^{-\lambda t} - e^{-(m^2\pi^2k/a^2)t})(-1)^{m+1}$$

$$\cdot\left[\frac{(-1)^m - \frac{r_1}{a}\cos\frac{m\pi r_2}{a} + \frac{1}{m\pi}\sin\frac{m\pi r_1}{a}}{m(a^2\lambda - m^2\pi^2k)}\right].$$

The behaviour of these solutions is quite complicated. Some results for the special case of the Earth are given in the references; [3] contains a discussion of the reasons for using this solution, and the method of applying it to fairly complicated distributions of radioactivity.

5. Application to the diffusion of argon. An interesting application of the above solution is to the problem of the diffusion of a gas being produced within a solid by radioactive decay.* The differential equation which applies is (using the notation for the case of the gas argon-40 being produced by the decay of potassium-40)

$$\frac{\partial A_c^{40}}{\partial t} = D\nabla^2 A_c^{40} + \lambda_k K_0^{40} e^{-\lambda t} \tag{40}$$

where A_c^{40} denotes the concentration of argon-40, K_0^{40} the initial concentration of potassium-40, and λ and λ_k are constants.

The solution required is that for the *average* argon content at time t (denoted by A^{40}) of a spherical solid with the potassium distributed homogeneously throughout, and with zero argon content at $t=0$ and zero surface concentration of argon (*i.e.*, similar boundary conditions to those of section 1). This solution is given by setting $r_1=0$ in (39), and substituting A^{40} for $\bar{\theta}$, D for k, and $\lambda_k K_0^{40}$ for J. Thus,

$$\frac{A^{40}}{\lambda_k K_0^{40}} = \frac{6a^2}{\pi^2}\sum_{m=1}^{\infty}\frac{e^{-\lambda t} - e^{-(m^2\pi^2D/a^2)t}}{m^2(m^2D\pi^2 - a^2\lambda)}, \tag{41}$$

or, writing K^{40} for $K_0^{40}e^{-\lambda t}$, c for λ_k/λ, and s for $a\sqrt{\lambda/D}$,

$$\frac{1}{c}\frac{A^{40}}{K^{40}} = \frac{6}{\pi^2}\sum_{m=1}^{\infty}\frac{1 - e^{-[(m^2\pi^2/s^2)-1]\lambda t}}{m^2\left(m^2\frac{\pi^2}{s^2} - 1\right)}. \tag{42}$$

The diagram shows the behaviour of this function for various values of s. (The

* This was pointed out to me by Dr. H. A. Shillibeer, who developed the results independently in connection with his researches on the potassium-argon method of dating minerals. The diagram and most of this section are taken from [4], with his kind permission.

λ used is the decay constant for potassium-40, $.55\times10^{-9}$ years^{-1}) Evidently there are two distinct types of curves. This can be seen from the formula (42), for

$$\lim_{t\to\infty}\left(\frac{1}{c}\frac{A^{40}}{K^{40}}\right)=\begin{cases}\dfrac{6}{\pi^2}\displaystyle\sum_{m=1}^{\infty}\frac{1}{m^2\left(\dfrac{m^2\pi^2}{s^2}-1\right)}=\dfrac{3}{s^2}-\dfrac{3}{s}\cot s-1 & \text{if } s<\pi,\\ +\infty & \text{if } s>\pi.\end{cases} \tag{43}$$

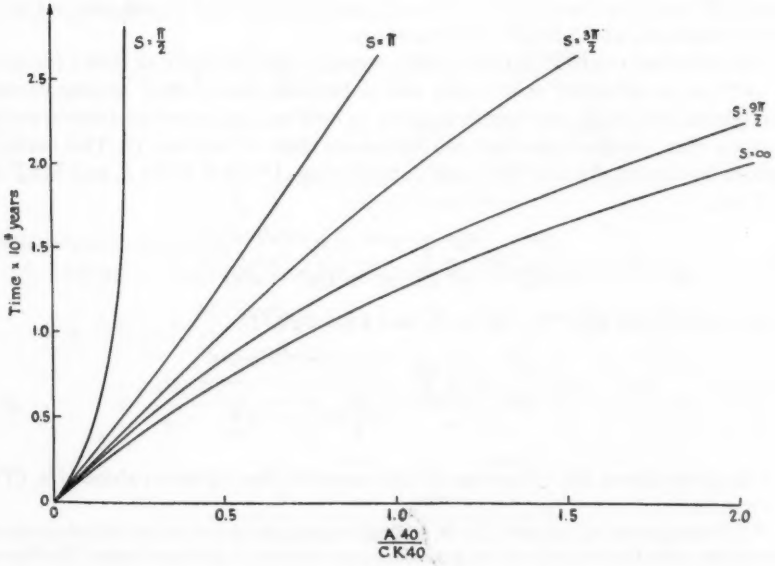
The limiting case $s=\pi$ is given by

$$\frac{1}{c}\frac{A^{40}}{K^{40}}=\frac{6\lambda t}{\pi^2}+\frac{6}{\pi^2}\sum_{m=2}^{\infty}\frac{1-e^{-(m^2-1)\lambda t}}{m^2(m^2-1)}, \tag{44}$$

which, as the graph shows, is asymptotic from below to a straight line, *viz.*

$$\frac{1}{c}\frac{A^{40}}{K^{40}}=\left(\frac{21}{2\pi^2}-1\right)+\frac{6\lambda}{\pi^2}t. \tag{45}$$

It may be noted that if s were $<\pi$ in physical fact, it would be almost impossible to use measurements of A^{40}/K^{40} to determine t, the age. However, experimental results obtained for argon seem to show s is large for most mineral samples, so that this age-determination method is practicable.



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MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

EVALUATION OF PRODUCT SEQUENCES BY MATRIX METHODS*

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1. Introduction. Let

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k \tag{1}$$

be a regular transformation by means of which a sequence $s_0, s_1, \cdots$, is evaluable to σ if the series in (1) converges for each n and $\sigma_n \to \sigma$ as $n \to \infty$. The hypothesis that (1) is regular means that $\sigma_n \to s$ whenever $s_n \to s$. We shall use the well known conditions

$$\sum_{k=0}^{\infty} |a_{nk}| \leqq M \qquad n = 0, 1, 2, \cdots \tag{2}$$

$$\lim_{n \to \infty} a_{nk} = 0 \qquad k = 0, 1, 2, \cdots \tag{3}$$

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 1, \tag{4}$$

which are necessary and sufficient for regularity of (1). An exposition of these matters is given, for example, by Hardy [1].

We shall say that the transformation (1) and the matrix $A = (a_{nk})$ have the product property P if A is real, regular, and such that if s_n and s_n' are real bounded sequences evaluable to σ and σ' respectively, then the product sequence $s_n s_n'$ is evaluable to the product $\sigma\sigma'$.

* A similar problem is considered by Mazur and Orlicz: On linear methods of summation, Studia Math., vol. 14, 1954, pp. 129–160.

2. Sequences of zeros and ones. The following theorem shows that the methods of Cesàro, Abel, and Borel and many other standard methods for evaluation of sequences fail to have property P. Moreover it shows that if A has property P, then relatively few sequences of zeros and ones are evaluable A.

THEOREM 1. *If A has property P and if $s_0, s_1, \cdots$, is a sequence of zeros and ones evaluable to σ, then σ is 0 or 1.*

To prove this theorem, we note that if A has property P and if s_n is a sequence of zeros and ones evaluable A to σ, then the sequence $s_n s_n$ (which is the same as the sequence s_n) is evaluable A to σ^2, so that $\sigma=\sigma^2$ and σ is 0 or 1.

We now employ some results of which an exposition has been given by Hill [2]. To each sequence $s_0, s_1, s_2, \cdots$ of zeros and ones there corresponds a point t of the interval $0 \leqq t \leqq 1$ having the binary expansion $s_0 s_1 s_2 \cdots$. If A is regular, then the set of sequences of zeros and ones evaluable A corresponds to a set of values of t having Lebesgue measure 0 or 1, and, moreover, if the measure is 1 then the set of zeros and ones evaluable A to $1/2$ corresponds to a set of values of t having measure 1. It follows from this and Theorem 1 that if A has property P, then the set of sequences of zeros and ones evaluable A corresponds to a set of values of t having measure 0.

3. A criterion for property P.

THEOREM 2. *A real regular matrix A has property P if and only if s_n^2 is evaluable A to σ^2 whenever s_n is a real bounded sequence evaluable A to σ.*

Let a real regular matrix have property Q if s_n^2 is evaluable A to σ^2 whenever s_n is a real bounded sequence evaluable A to σ. Since we may take $s_n'=s_n$, it is obvious that if A has property P then it also has property Q. To prove the converse, let A have property Q and let s_n and s_n' be real bounded sequences evaluable A to σ and σ' respectively. The real bounded sequences $\{s_n+s_n'\}$ and $\{s_n-s_n'\}$ are then evaluable A to $\sigma+\sigma'$ and $\sigma-\sigma'$ and hence the sequences $\{(s_n+s_n')^2\}$ and $\{(s_n-s_n')^2\}$ are evaluable A to $(\sigma+\sigma')^2$ and $(\sigma-\sigma')^2$. Since $4s_n s_n'=(s_n+s_n')^2-(s_n-s_n')^2$ and $4\sigma\sigma'=(\sigma+\sigma')^2-(\sigma-\sigma')^2$, it follows that $s_n s_n'$ is evaluable A to $\sigma\sigma'$ and Theorem 2 is proved. A variant of Theorem 2 which may be useful is the following.

THEOREM 3. *A real regular matrix A has property P if and only if s_n^2 is evaluable A to 0 whenever s_n is evaluable A to 0.*

This is a consequence of the fact that if s_n^2 is evaluable A to 0 whenever s_n is evaluable A to 0 and if s_n' is evaluable A to σ' then $\{s_n'-\sigma\}$ is evaluable A to 0, so $\{(s_n'-\sigma)^2\}$ is evaluable A to 0, and linearity of A implies that $(s_n')^2$ is evaluable A to $(\sigma')^2$.

4. Consequences of property P.

THEOREM 4. *If a real bounded sequence s_n is evaluable to σ by a matrix having property P, then*

$$\liminf_{n\to\infty} |s_n| \leqq |\sigma| \leqq \limsup_{n\to\infty} |s_n|. \tag{5}$$

To prove Theorem 4, it will be sufficient to prove that

$$\limsup_{n\to\infty} |s_n^2 - x| \geqq |\sigma^2 - x|, \qquad x \geqq 0, \tag{6}$$

because putting $x=0$ in (6) gives the last inequality in (5), and choosing x so great that $x>\sigma^2$ and $x>s_n^2$ for each n gives the first inequality in (5). To prove (6), suppose that there is some real x for which it is false. Then $x\neq\sigma^2$ and there is a number θ for which

$$\limsup_{n\to\infty} \left|\frac{s_n^2 - x}{\sigma^2 - x}\right| < \theta < 1. \tag{7}$$

With M satisfying (2), choose a positive integer m such that $\theta^m M<1$. Let

$$t_n = \frac{s_n^2 - x}{\sigma^2 - x}. \tag{8}$$

The hypothesis of Theorem 4 implies that s_n^2 is evaluable A to σ^2 and hence that s_n^2-x is evaluable A to σ^2-x so that t_n is evaluable A to 1 and t_n^m is evaluable A to 1. Thus

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} t_k^m = 1. \tag{9}$$

Because of (7) and (8), we can choose k_0 such that $|t_k| <\theta$ when $k\geqq k_0$. Use of (9) and (3) then gives

$$\lim_{n\to\infty} \sum_{k=k_0}^{\infty} a_{nk} t_k^m = 1. \tag{10}$$

But use of (2) gives, for each $n=0, 1, 2, \cdots$,

$$\left|\sum_{k=k_0}^{\infty} a_{nk} t_k^m\right| \leqq \theta^m \sum_{k=k_0}^{\infty} |a_{nk}| \leqq \theta^m M < \frac{1}{2}. \tag{11}$$

The contradiction in (10) and (11) establishes (6) and completes the proof of Theorem 4.

THEOREM 5. *If a real bounded sequence s_n is evaluable to σ by a matrix A having property P, then σ is a cluster value (or limit point) of the sequence s_n.*

The hypothesis implies that the sequence $s_0-\sigma$, $s_1-\sigma$, $\cdots$ is evaluable A to 0 and hence Theorem 4 implies that

$$\liminf_{n\to\infty} |s_n - \sigma| \leqq 0 \leqq \limsup_{n\to\infty} |s_n - \sigma|.$$

The result follows.

5. Conclusion. For the following remarks, the author is indebted to R. P. Agnew. The transformation defined by $\sigma_0=s_0$, $\sigma_1=s_1$, and

$$(12)\qquad \begin{cases} \sigma_{2n} = -2s_{2n-1} + 3s_{2n} & n > 0, \\ \sigma_{2n+1} = -4s_{2n-1} + 5s_{2n} & n > 0, \end{cases}$$

has property P. This is an easy consequence of the fact that $\sigma_n \to L$ if and only if $s_{2n} \to L$ and $s_{2n-1} \to L$, the values of the numbers s_{2n-2} being completely irrelevant. It may be possible to show that each transformation having property P must have a form resembling (12) in essential respects.

The following seems to be an entirely reasonable conjecture. A given matrix and transformation A have property P if and only if there is an increasing sequence $n_1, n_2, n_3, \cdots$ of non-negative integers which depends only upon A and which is such that an arbitrary sequence $s_0, s_1, s_2, \cdots$ is evaluable to L if and only if the special subsequence $s_{n1}, s_{n2}, s_{n3}, \cdots$ converges to L.

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RATIONAL UNIVALENT FUNCTIONS

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Let $\mathfrak{F}$ be the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are regular and univalent in the unit circle $E(|z|<1)$. The subclasses of $\mathfrak{F}$ whose coefficients a_n belong to a quadratic field have been studied by Friedman [1] and Bernardi [2]. Recently Linis [3] gave a short proof of a theorem of Friedman which states that if the a_n are rational integers then $f(z)$ is a rational function and is one of the nine functions:

$$z,\ z/(1\pm z),\quad z/(1\pm z^2),\quad z/(1\pm z)^2,\quad z/(1\pm z+z^2).$$

He also gave some rational functions whose expansion had Gaussian integers as coefficients.

The purpose of this note is to extend the method of proof given by Linis to quadratic fields with negative discriminant and to give criteria for selecting which combinations of possible coefficients yield rational functions which are regular and univalent in E.

For completeness we will sketch Linis' proof. Let a_n be a rational integer. Let $z/f(z)=\sum_{n=0}^{\infty} b_n z^n$. By Prawitz's theorem [4] $\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leqq 1$. Clearly $b_0=1$, $b_1=-a_2$, and since $f(z)$ is univalent in E, $|a_2| \leqq 2$ so that $|b_1| \leqq 2$. From Prawitz's theorem it follows that $|b_n| \leqq 1$, $n \geqq 2$. The coefficients b_n can be computed by the determinant $(-1)^n b_n=\Delta(C_{ij})$ where $C_{ij}=0$, $j>i+1$; $C_{ij}=1$, $j=i+1$; $C_{ij}=a_{j-i+2}$, $i, j=1, \cdots, n$. Hence the b_n are rational integers. The

possible values of b_n are: $b_0=1$; $b_1=0, \pm 1, \pm 2$; $b_2=0, \pm 1$; $b_n=0$, $n \geqq 3$. Certain combinations of these values yield the nine above-mentioned functions.

Let $R(\sqrt{m})$ be a quadratic field where m is a square-free rational integer. If $m \equiv 2$ or $3 \pmod 4$ the integers of $R(\sqrt{m})$ are given by $a+b\sqrt{m}$, where a and b range over all rational integers. If $m \equiv 1 \pmod 4$ the integers of $R(\sqrt{m})$ are given by $\frac{1}{2}(a+b\sqrt{m})$ where a and b are both even or both odd rational integers. We will be interested only in the cases in which m is one and in which m is negative. The units of $R(\sqrt{m})$ are those numbers whose norm is ± 1. The only units of $R(\sqrt{m})$ where m is negative are ± 1 unless $m=-1, -3$, in which case the additional units are $\pm i$ and $\pm\omega, \pm\bar{\omega}$, respectively, where $\omega=\frac{1}{2}(-1+\sqrt{3}i)$ and $\bar{\omega}$ is the complex conjugate of ω. (For the arithmetic of algebraic number fields see [5], Chap. VII).

Let $a_n \in R(\sqrt{m})$, for a given m, the b_n are given by the previously mentioned determinant, so that $b_n \in (R\sqrt{m})$. Since $R(\sqrt{m})$ has no elements whose modulus is a number between zero and one, it follows from Prawitz's theorem that b_n, $n \geqq 3$, in the expansion of $z/f(z)$ are all zero.

The possible values of b_n are as follows:

$$m=-1; \quad b_0=1; \quad b_1=0, \pm 1, \pm 2, \pm i, \pm 2i, \pm(1 \pm i); \quad b_2=0, \pm 1, \pm i.$$

$$m=-2; \quad b_0=1; \quad b_1=0, \pm 1, \pm 2, \pm\sqrt{2}i, 1 \pm 2i, \quad b_2=0, \pm 1.$$

$$m=-3; \quad b_0=1; \quad b_1=0, \pm 1, \pm 2, \pm\omega, \pm\bar{\omega}, \pm 2\omega, \pm 2\bar{\omega}, \pm(1-\omega), \pm(1-\bar{\omega}), \pm(\omega-\bar{\omega}); \quad b_2=0, \pm 1, \pm\omega, \pm\bar{\omega}.$$

$m<-3$; $b_0=1$, $b_1=0, \pm 1, \pm 2$; $b_2=0, \pm 1$, the same as when $m=1$.

Before enumerating the b_n which yield regular univalent rational functions, let us state two lemmas.

LEMMA 1. *The function $f(z)=z/(1+b_1z)$ is regular and univalent in E if and only if $|b_1| \leqq 1$.*

LEMMA 2. *The function $f(z)=z/(1+b_1z+b_2z^2)$ with $|b_2|=1$ is regular and univalent in E if and only if $b_1/(b_2)^{1/2}$ is real and in modulus less than or equal to 2.*

The proof of Lemma 1 is clear. The univalency in Lemma 2 follows immediately. Since $|b_2|=1$, $f(z)$ will be regular in E if and only if the zeros of the denominator lie on the periphery of E. Let $z=s/(b_2)^{1/2}$, then the denominator is $s^2+[b_1/(b_2)^{1/2}]s+1$. Its zeros are on $|s|=1$ and hence are complex conjugates. Using these properties and the quadratic formula we see that the zeros will be of unit modulus if and only if b_1^2/b_2 is positive and less than or equal to 4 which implies that $b_1/(b_2)^{1/2}$ is real and in modulus less than or equal to 2.

We note that every field $R(\sqrt{m})$ includes the field $R(\sqrt{1})$, hence we will enumerate those b_n not concerned with $R(\sqrt{1})$.

Making use of the following facts: $(\omega)^{1/2}=-\bar{\omega}$, $(-\omega)^{1/2}=i\bar{\omega}$, $(\bar{\omega})^{1/2}=\omega$, $(-\bar{\omega})^{1/2}=-i\omega$, and the properties of the cube roots of unity, we can easily construct a table of values of the b_n which satisfy the two lemmas.

b_1	0	1	-1	i	$-i$	ω	$-\omega$	$\bar{\omega}$	$-\bar{\omega}$
b_1, $m=-1$	$\pm i$	none	$\pm i$, $\pm 2i$	0, $\pm(1+i)$	0, $\pm(1-i)$				
b_1, $m=-2$	none	none	$\pm\sqrt{2}i$	none	none				
b_1, $m=-3$	$\pm\omega$, $\pm\bar{\omega}$	none	$\pm(\omega-\bar{\omega})$	none	none	0, $\pm\bar{\omega}$, $\pm 2\bar{\omega}$	0, $\pm(1-\omega)$	0, $\pm\omega$, $\pm 2\omega$	0, $\pm(1-\bar{\omega})$
b_1, $m<-3$	Same as when $m=1$.								

This proves the following theorem first given by Bernardi [2] by other methods.

THEOREM. *The subclass of $\mathfrak{F}$ having algebraic integral coefficients in $R(\sqrt{m})$, $m=1$ or a square-free rational negative integer, consists of forty five rational functions of the form $f(z)=z/(1+b_1z+b_2z^2)$, where b_1 and b_2 are given by the above table.*

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A CONJECTURE CONCERNING NEIGHBORING TETRAHEDRA

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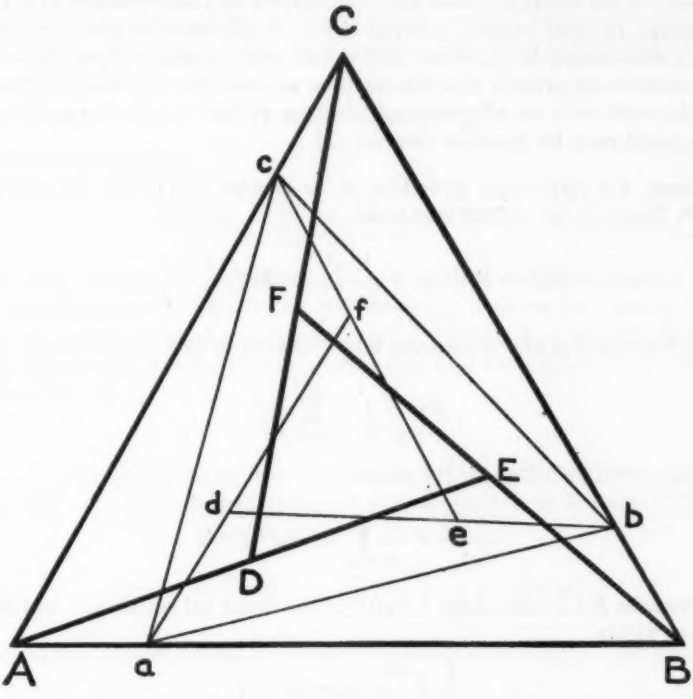
Two bodies in space are said to be neighboring if they have no interior points in common and there is a surface belonging to both their frontiers. A (finite or infinite) number of bodies are said to be neighboring if every pair of them are neighboring.

It is known that it is possible to have infinitely many neighboring bodies—even infinitely many neighboring convex bodies.* This suggests the problem of defining, by means of some property, a class of bodies such that the maximum possible number of neighboring bodies of this class is finite, and of determining this maximum number.

Consider the class of tetrahedra. It is not difficult to verify that at most 4 neighboring tetrahedra can stand with their bases on one and the same plane (and lie, with the exception of these bases, on one and the same side of this plane). Consequently, the maximum possible number of neighboring tetrahedra

* For the literature on this result, as well as an interesting discussion of some related problems, see Heinrich Tietze, Gelöste und ungelöste mathematische Probleme aus alter und neuer Zeit, Munich, Biederstein, 1949.

is certainly not greater than 17, and we conjecture that it is 8. That it is at least 8 is shown by the following example.



In the accompanying figure, all the lines are supposed to be coplanar. Let Q and q be points on opposite sides of the plane ABC. Then the following 8 tetrahedra are obviously neighboring:

$$QABE, \quad QBCF, \quad QCAD, \quad QDEF,$$
$$qabd, \quad qbce, \quad qcaf, \quad qdef.$$

ON ANALYTIC FUNCTIONS HAVING A POSITIVE REAL PART IN THE UNIT CIRCLE

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A theorem of basic importance in the theory of univalent functions is the following:

THEOREM. *Let* $f(z)=1+\sum_{n=1}^{\infty} a_n z^n$ *be regular and have* $\mathrm{Re}(f(z))^* \geqq 0$ *in* $|z|<1$. *Then* $|a_n| \leqq 2$, $n=1, 2, \cdots$.

* By $\mathrm{Re}(f(z))$ we understand the real part of $f(z)$.

The function $f(z)=(1+z)(1-z)^{-1}$ shows that this bound on the coefficients is sharp.

Proofs of the above theorem have been given by Carathéodory [1], Toeplitz [4], Nehari [3] and others. It is the object of this note to give another proof which is elementary in character and would seem to suggest from its structure the theorem to be proven. For this purpose we state the following lemma whose proof depends only on simple manipulations of the Cauchy integral formula. Such a proof may be found in Goursat [2].

LEMMA. *Let* $f(z)=U(x, y)+iV(x, y)$ *be regular in* $|z|\leqq R$. *Let* $z=re^{i\theta}$ *and* $\zeta=Re^{i\psi}$. *Then for* $|z|<R$ *we may write*

$$f(z) = iV(0, 0) + \frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\psi})\frac{\xi+z}{\xi-z}\,d\psi. \tag{1}$$

The proof of the above theorem now proceeds as follows. We have

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n,$$

where $a_n=f^{(n)}(0)/n!$. But by (1) above

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{\pi}\int_0^{2\pi} U(Re^{i\psi})\xi^{-n}d\psi$$

where we take $R<1$. Also since $U(0, 0)=1$ the mean value theorem for harmonic functions yields

$$\frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\psi})d\psi = 1.$$

But since $U(x, y)\geqq 0$ it follows that

$$R^n|a_n| \leqq 2\left\{\frac{1}{2\pi}\int_0^{2\pi}|U(Re^{i\psi})|\,d\psi\right\} = 2\left\{\frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\psi})d\psi\right\} = 2\cdot 1 = 2$$

for all $R<1$ and therefore $|a_n|\leqq 2$, $n=1, 2, \cdots$.

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ON THE nTH DERIVATIVE OF COMPOSITE FUNCTIONS

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Let f be the function whose value at x is $f(x)$. Let $\mathbf{D}^n f$ denote the nth derivative of f, that is, the function assuming for any x the value $\mathbf{D}^n f(x)$. If f and g are two functions, then

$$\mathbf{D}^n(f+g) = \mathbf{D}^n f + \mathbf{D}^n g \tag{1_n}$$

and according to Leibniz

$$\mathbf{D}^n(f\cdot g) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!}\mathbf{D}^{n-r}f\cdot\mathbf{D}^r g. \tag{2_n}$$

Let f and g be functions. Throughout this note the juxtaposition of two functions will denote the substitution of the second into the first (hence, the dot will not be omitted in products of functions). For example, $(fg)(x) = (fg(x))$ and $(\mathbf{D}f)g$ is the result of substituting g into $\mathbf{D}f$. The derivative of the function fg is then given by:

$$\mathbf{D}(fg) = (\mathbf{D}f)g\cdot\mathbf{D}g. \tag{3_1}$$

Formulae for $\mathbf{D}^n(fg)$ have been found by Faà di Bruno and Schlömilch.† The purpose of this note is the establishment of two other formulae for $\mathbf{D}^n(fg)$. The first is

$$\mathbf{D}^n(fg) = \sum_{r=1}^{n} (\mathbf{D}^r f)g\cdot \sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!}\, g^{r-s}\cdot\mathbf{D}^n(g^s). \tag{3_n}$$

Formula (3_n) gives an explicit form for the operators $\mathbf{Y}_{n,r}$ in the relation

$$\mathbf{D}^n(fg) = \sum_{r=1}^{n} (\mathbf{D}^r f)g\cdot\mathbf{Y}_{n,r}(g) \tag{4_n}$$

which was used by Schlömilch. It is known that the $\mathbf{Y}_{n,r}$ satisfy the recursion formula:§

$$\mathbf{Y}_{n+1,r}(g) = \mathbf{D}g\cdot\mathbf{Y}_{n,r-1}(g) + \mathbf{D}\mathbf{Y}_{n,r}(g), \tag{5}$$

where

$$\mathbf{Y}_{n,0}(g) = 0 = \mathbf{Y}_{n,n+1}(g). \tag{6}$$

Let f be the function assuming at x the value $f(x) = x^k/k!$, where $0 \leq k \leq n$. Substituting this f in (4_n) yields a second recursion formula

† See, for example, Charles Jordan, Calculus of Finite Differences, p. 33, Sopron, Hungary, 1939.

§ Riordan, Derivatives of composite functions, Bull. Am. Math. Soc., vol. 52, pp. 664–667, 1946. Also C. Jordan.

$$(7_{n,k}) \qquad \mathbf{Y}_{n,k}(g) = \frac{1}{k!}\mathbf{D}^n(g^k) - \sum_{r=0}^{k-1} \frac{g^{k-r}}{(k-r)!} \cdot \mathbf{Y}_{n,r}(g).$$

To prove (3_n), one must show that

$$(8_{n,r}) \qquad \mathbf{Y}_{n,r}(g) = \sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-s} \cdot \mathbf{D}^n(g^s).$$

The proof is by induction. By (3), $\mathbf{Y}_{1,1}(g) = \mathbf{D}g$. By induction, in view of (6) and (5), $\mathbf{Y}_{n,1}(g) = \mathbf{D}^n g$, which agrees with $(8_{n,1})$. Assume $(8_{n,k})$ for $k = 0, 1, \cdots, r-1$. To prove $(8_{n,r})$, in view of $(7_{n,r})$, one has to show

$$\sum_{s=0}^{r} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-s} \cdot \mathbf{D}^n(g^s) = \frac{1}{r!}\mathbf{D}^n(g^r) - \sum_{k=0}^{r-1} \frac{g^{r-k}}{(r-k)!} \cdot \sum_{s=0}^{k} \frac{(-1)^{k-s}}{s!(k-s)!} g^{k-s} \cdot \mathbf{D}^n(g^s).$$

The term for $s = r$ on the left equals $\mathbf{D}^n(g^r)/r!$ on the right. For a fixed s, $0 \leqq s \leqq r-1$, the coefficient of $g^{r-s} \cdot \mathbf{D}^n(g^s)$ on the right is

$$-\sum_{k=s}^{r-1} \frac{(-1)^{k-s}}{(r-k)!s!(k-s)!}.$$

Let $k = s + p$, then this becomes

$$-\frac{1}{s!} \sum_{p=0}^{r-s-1} \frac{(-1)^p}{(r-s-p)!p!},$$

which equals

$$-[(1-1)^{r-s} - (-1)^{r-s}]/[s!(r-s)!] = \frac{(-1)^{r-s}}{s!(r-s)!}.$$

But this is also the coefficient of $g^{r-s} \cdot \mathbf{D}^n(g^s)$ on the left. This proves $(8_{n,r})$.

The second formula for $\mathbf{D}^n(fg)$ is

$$(9_n) \qquad \mathbf{D}^n(fg) = \sum_{r=1}^{n} (\mathbf{D}^r f) g \cdot \frac{1}{r!} \sum \frac{n!}{p_1! \cdots p_r!} (\mathbf{D}^{p_1} g) \cdot \cdots \cdot (\mathbf{D}^{p_r} g)$$

where the second summation is taken over the integral solutions of $p_1 + \cdots + p_r = n$ and $p_i > 0$, for $i = 1, \cdots, r$. The proof of (9_n) is based on the following.

LEMMA: $\mathbf{Y}_{n,r}(g)$ *consists of those terms in the expansion of* $\mathbf{D}^n(g^r)/r!$ *which do not contain* g *as factor.*

By (3), $\mathbf{Y}_{1,1}(g) = \mathbf{D}g$. Assume that $\mathbf{Y}_{n,r}(g)$, $r = 0, 1, \cdots, n$ contains no terms with g as factor. It follows by (5) that $\mathbf{Y}_{n+1,r}(g)$ contains no terms with g as factor. But by $(8_{n,r})$,

$$Y_{n,r}(g) = g \cdot \sum_{s=0}^{r-1} \frac{(-1)^{r-s}}{s!(r-s)!} g^{r-1-s} \cdot D^n(g^s) + \frac{1}{r!} D^n(g^r).$$

Hence the first term on the right, containing g as factor, must cancel with the corresponding terms in the expansion of $D^n(g^r)/r!$.

The proof of (9_n) is completed by applying the multinomial analogue of (2_n) to $D^n(g^r)$.

$$\frac{1}{r!} D^n(g^r) = \frac{1}{r!} \sum \frac{n!}{p_1! \cdots p_r!} D^{p_1} g \cdot \cdots \cdot D^{p_r} g,$$

where the summation extends over the non-negative integral solutions of $p_1 + \cdots + p_r = n$. A factor g appears whenever one of the derivatives is of order 0. The stipulation $p_i > 0$ eliminates these terms, yielding, along with the lemma, formula (9_n).

In conclusion, we note the following consequences of the above formulae. In view of (5) it follows by induction that $Y_{n,n}(g) = [Dg]^n$ for any $n \geqq 1$. Hence, $(8_{n,n})$ yields the following identity

$$[Dg]^n = \sum_{s=0}^{n} \frac{(-1)^{n-s}}{s!(n-s)!} g^{n-s} \cdot D^n(g^s) \text{ for any } g. \tag{10}$$

It may also be shown that

$$Y_{n,n-1}(g) = \frac{n(n-1)}{2} \cdot D^2 g \cdot [Dg]^{n-2}.$$

Hence:

$$D^2 \cdot g[Dg]^{n-1} = \frac{2(n-1)!}{(n+1)!} \sum_{s=0}^{n} \frac{(-1)^{n-s}}{(n-s)!s!} g^{n-s} \cdot D^{n+1}(g^s) \text{ for any } g. \tag{11}$$

Also, comparing (9_n) with the corresponding formula of Faà di Bruno, we obtain the identity

$$\sum \frac{\left(\frac{Dg}{1!}\right)^{q_1} \cdot \left(\frac{D^2 g}{2!}\right)^{q_2} \cdot \cdots \cdot \left(\frac{D^n g}{n!}\right)^{q_n}}{q_1! \cdot q_2! \cdot \cdots \cdot q_n!} = \frac{1}{r!} \sum \frac{(D^{p_1} g) \cdot \cdots \cdot (D^{p_r} g)}{p_1! \cdot \cdots \cdot p_r!} \tag{12}$$

where the summation is over

$$q_1 + q_2 + \cdots + q_n = r$$

and

$$q_1 + 2q_2 + \cdots + nq_n = n$$

where the summation is over

$$p_1 + p_2 + \cdots + p_r = n$$

and

$$p_i > 0.$$

Many other identities may be found by particular choices of g in (10), (11), and (12).

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A FUNDAMENTAL THEOREM OF CALCULUS

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The following theorem is frequently called the fundamental theorem of the differential calculus.

THEOREM A. *If F_1 and F_2 are functions which have the same derivative at each point of an interval I, then there exists a constant c such that $F_1(x)-F_2(x)=c$ for each x in I.*

The proof of Theorem A follows easily from:

THEOREM B. *If F is a function whose derivative is 0 at each point of an interval I, then there exists a constant c such that $F(x)=c$ for each x in I.*

The usual proof of Theorem B is based on the Mean Value Theorem, the proof of the Mean Value Theorem is based on Rolle's Theorem, and proof of Rolle's Theorem is based on the Maximum Value Theorem for continuous functions. Since a proof of this last theorem is often omitted from the first course in calculus, a complete proof of Theorem B is usually not given. Recently, Professor Harry D. Ruderman pointed out to the author that it would be desirable to have a direct proof of Theorem B which would avoid the use of the Mean Value Theorem. In this note we give such a proof.

We start with the fact that every bounded, monotone sequence of real numbers converges. This fact is stated as an axiom in many calculus texts. It is then easy to prove that if $J_1 \supset J_2 \supset J_3 \supset \cdots$ is a nested sequence of closed intervals which is obtained by the "bisection process", then there exists a unique number p which is a member of each J_n. Moreover, if x_n is a member of J_n for each n, then the sequence $x_1, x_2, x_3, \cdots$ converges to p. We now give a proof of Theorem B which is based on these facts about nested sequences of closed intervals.

It is sufficient to prove that if a and b are points of I then $F(a)=F(b)$. We may assume that $a<b$.

If $J=[u, v]$ is a closed interval which is contained in I, we define $S(J)=\left|(F(u)-F(v))/(u-v)\right|$. The following lemma is obvious and its proof is omitted.

LEMMA. *If $u<t<v$, $J'=[u, t]$, $J''=[t, v]$ and $J=[u, v]$, then either $S(J') \geqq S(J)$ or $S(J'') \geqq S(J)$.*

We make use of the Lemma and the "bisection process" to obtain a sequence $J_1 \supset J_2 \supset J_3 \supset \cdots$ of closed intervals such that $J_1=[a, b]$ and $S(J_{n+1}) \geqq S(J_n)$

for each n. There exists a unique point p which is a member of J_n for each n. We now use the Lemma to obtain for each n a non-degenerate subinterval K_n of J_n such that one end point of K_n is p, the other end point x_n of K_n is also an end point of J_n, and $S(K_n) \geqq S(J_n)$. The sequence $x_1, x_2, x_3, \cdots$, converges to p, and since $F'(p)=0$ we obtain $\lim_{n\to\infty} (F(x_n)-F(p))/(x_n-p)=0$. Thus $\lim_{n\to\infty} S(K_n)=0$. Since $S(K_n) \geqq S(J_n) \geqq S(J_1) \geqq 0$ for each n, we see that $S(J_1)=0$. This implies that $F(a)=F(b)$, which concludes the proof of Theorem B.

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS*

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Let $\{y_i\}$ $(i=1, \cdots, n)$ be a set of independent variables, and suppose that a system of m linear homogeneous differential equations with constant coefficients is to be solved. The usual approach to this problem is to introduce derivatives of the $\{y_i\}$ as auxiliary variables, reducing the system to a (larger) set of equations all of the first order. In the following, an alternative approach is given, which makes use of the "elementary divisor theory" for matrices of polynomials.

Using matrix notation, the system of m equations may be written

$$T(D)Y = 0, \tag{1}$$

where $T(D)$ is an m by n matrix whose elements are polynomials in the differential operator D, and Y is a column vector whose components are the y_i.

A natural approach to the solution of (1), analogous to that of the theory of a single linear equation, is to assume a solution of form

$$Y = (K_0 + K_1 x + \cdots + K_s x^s)e^{ax}, \tag{2}$$

where the K_i are constant vectors and a is a scalar constant. Substituting in (1) and equating coefficients of x to zero, we obtain a series of equations

$$\begin{aligned}
&T(a)K_s = 0 \\
&T(a)K_{s-1} + sT'(a)K_s = 0 \\
&T(a)K_{s-2} + (s-1)T'(a)K_{s-1} + \frac{s(s-1)}{2!}T''(a)K_s = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&T(a)K_0 + T'(a)K_1 + T''(a)K_2 + \cdots + T^{(s)}(a)K_s = 0.
\end{aligned} \tag{3}$$

It is clear from the first equation that a necessary condition for the existence of a solution is that $T(a)$ must be singular; that is, the equation $T(a)X=0$ must be solvable. (If $T(a)$ is square, for example, this means that a must be a root of the characteristic equation $|T(x)|=0$.)

* Presented (in part) to the Mathematical Association of America, Nebraska Section, May 2, 1953.

In some cases† it is convenient to solve these equations iteratively for K_s, K_{s-1}, $\cdots$. However this procedure leads in general to certain difficulties such as arise, for example, when the first equation of (3) has several solutions not all of which make possible a solution of the second equation, and so on. Furthermore, it is not clear from this approach just what value of s should be used, nor is it evident that all solutions must necessarily be of form (2).

To analyze this problem more carefully we take advantage of the isomorphism* between the polynomials in D and the set of all polynomials in an indeterminate λ. This makes possible the use of a well-known theorem which states that there exist square matrices $P(D)$ and $Q(D)$ each having unit determinant such that $U=PTQ$ has form

$$U(D) = \begin{pmatrix} u_1 & & & & & \\ & u_2 & & & 0 & \\ & & \ddots & & & \\ & & & u_h & & \\ & & & & 0 & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \tag{4}$$

in which the rank h is the same as that of $T(D)$ and the $\{u_i\}$ (the "invariant factors") are so arranged that u_1 divides u_2, u_2 divides u_3, and so on.§ The matrices $P(D)$ and $Q(D)$ may, in fact, be obtained as products of elementary matrices. Thus substitution in (1) of $Y=QX$ and multiplication by P leads to the equation

$$UX = 0. \tag{5}$$

The first result is that if $h<n$ there will be $n-h$ solutions of (5) involving arbitrary functions. These will be of form

$$X_i = \psi_i(x)\delta_i \qquad (i = h+1, \cdots, n) \tag{6}$$

with ψ_i arbitrary and δ_i the vector having 1 in the ith position, zero elsewhere. Using the usual properties of the differential operator, it is easily verified that (1) has corresponding solutions

$$Y_i = \psi_i Q_{i0} + \psi_i' Q_{i0}' + \frac{\psi_i'' Q_{i0}''}{2!} + \cdots \qquad (i = k+1, \cdots, n), \tag{7}$$

† See Frazer, Duncan and Collar, Elementary Matrices, Cambridge University Press, 1952, p. 172.

* For simplicity, we will henceforth use this isomorphism without explicit reference. For example, when we say that $g(D)$ is a factor of $f(D)$ we will mean that $g(\lambda)$ divides $f(\lambda)$.

§ See, for example, C. C. MacDuffee, Vectors and Matrices, Carus Mathematical Monograph No. 7, Math. Assoc. of Am., 1943, p. 143. (The generalization to non-square matrices is obvious.)

where $Q_{i0}^{(k)}$ indicates the ith column of the kth derivative of $Q(\lambda)$ evaluated at $\lambda=0$.

Now suppose that one of the invariant factors, say u_i, has a factor $(D-a)^c$ (an "elementary divisor"). Then clearly (5) has solutions

$$X_j = t^j e^{at}\delta_i \qquad (j = 0, 1, \cdots, c-1). \tag{8}$$

Again using well-known properties of the differential, the corresponding solutions of (1) are

$$Y_j = (t^j Q_{i0} + jt^{j-1}Q_{i0}' + \frac{j(j-1)}{2!}t^{j-2}Q_{i0}'' + \cdots + Q_{i0}^{(j)})e^{at} \tag{9}$$

where $Q_{i0}^{(j)}$ is the ith column of the jth derivative of $Q(\lambda)$ evaluated at $\lambda=a$. Note that this is a solution of type (2) with $s=j$.

Now it is clear from an examination of the form of $U(D)$, (4), that the only solutions of (5) (other than those given in (6)) are of this type. Then since every solution of (1) corresponds to a solution of (3), it follows that the set of all solutions of type (9), together with (7), give all solutions of (1). Hence a knowledge of the elementary divisors of $T(D)$, and of the transformation matrix $Q(D)$ permits us to write down immediately the general solution of the system (1).

ON FINDING DIAMETERS OF CONIC SECTIONS

E. A. HEDBERG, University of South Carolina

Nearly all current textbooks in analytic geometry give special formulas for diameters of conic sections in which the chords have a given slope m, listing each type of conic separately and giving only the standard forms, where the center or vertex is at the origin. That all these formulas are related seems, from this writer's experience, not universally known. I state the general case as a theorem.

THEOREM. *Let $f(x, y)\equiv Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ be the general equation of a conic section. The equation of the diameter which bisects the chords of slope m may be found by differentiating $f(x, y)=0$ with respect to x and replacing dy/dx by m.*

The statement of the theorem implies the existence of chords with slope m. It will be pointed out in the proof of the theorem that the condition that chords exist is that $A+Bm+Cm^2\neq 0$. Attention is also called to the fact that the theorem applies to the degenerate conics.

One may establish the theorem by direct substitution of $y=mx+k$ in $f(x, y)=0$, setting up the formulas for midpoints of the chords, eliminating the parameter k from these expressions, and comparing the result with the theorem stated above. However, a proof which avoids such manipulations would seem desirable. The following proof involves a minimum of actual computation.

The fact that the locus of the midpoints of the chords with a given slope m is a straight line may be verified without determining the actual locus. For if one substitutes $mx+k$ for y in the expanded form of $f(x, y)$, in order to determine the abscissas of the points of intersection of the line $y=mx+k$ with the conic $f(x, y)=0$, it is at once apparent that k does not enter $A+Bm+Cm^2$, the coefficient of x^2, and that it enters the coefficient of x to the first degree only. If $A+Bm+Cm^2 \neq 0$ the quadratic equation has two roots, x_1 and x_2, and the abscissa, $X=\frac{1}{2}(x_1+x_2)$, of the midpoint of the chord involves k to the first degree only. The ordinate, Y, of the midpoint, found from $Y=mX+k$, is linear in k, hence it is plain that the equation obtained by eliminating k from the expressions for X and Y is linear in X and Y. This straight line will henceforth be referred to as the diameter.

The coefficients which appear in the equation of the diameter are rational functions of m and the coefficients of $f(x, y)$. The only denominators which appear are constant multiples of $A+Bm+Cm^2$, hence we may express the equation of the diameter using polynomial coefficients by clearing of fractions. The method used in obtaining the equation is obviously independent of the type of conic under consideration, the only limitation being that $A+Bm+Cm^2$ is not zero.

If $A+Bm+Cm^2=0$ then there is at most one finite point of intersection of the line $y=mx+k$ with $f(x, y)=0$, whatever value of k is used. That is, in this case, there are no chords, in the ordinary sense, for the corresponding value of m. The parabola, for instance, has no chords parallel to its axis, the hyperbola none parallel to an asymptote. In the following, therefore, it will be assumed that $A+Bm+Cm^2 \neq 0$.

We now find the equation of the diameter, using the expanded form of $f(x, y)$ without making any explicit substitutions. If $mx+k$ were substituted for y in $f(x, y)=0$, and the condition for equal roots applied, which is also the condition that $y=mx+k$ be tangent to $f(x, y)=0$, the result would be $(d/dx)f(x, mx+k)=0$. However,

$$\frac{d}{dx} f(x, mx + k) \equiv \frac{\partial}{\partial x} f(x, y)\Big|_{y=mx+k} + m \frac{\partial}{\partial y} f(x, y)\Big|_{y=mx+k}.$$

That is, the points $(x, mx+k)$ at which the lines of the system $y=mx+k$ are tangent to the conic $f(x, y)=0$ lie on the straight line

$$\frac{\partial}{\partial x} f(x, y) + m \frac{\partial}{\partial y} f(x, y) = 0.$$

Now for an ellipse there are always two real points of tangency. The diameter plainly passes through these points, hence the above process leads to the equation of the diameter. But we have seen that the coefficients in the equation of the diameter are independent of the type of conic involved. Thus, in every case, the differentiation process leads to the equation of the diameter.

An application of the above process to finding the equations of the medians of a triangle may be of interest. The following represents the solution in the general case. Let $L_1 \equiv A_1x+B_1y+C_1=0$, $L_2 \equiv A_2x+B_2y+C_2=0$, $L_3 \equiv A_3x+B_3y+C_3=0$ be the equations of the sides of a triangle. Let us assume that $B_3 \neq 0$. To find the equation of the median upon the side $L_3=0$ write the equation of the diameter of the degenerate conic $L_1L_2=0$ which bisects the chords of slope $-A_3/B_3$. Applying the general rule of the theorem above, one arrives at the result:

$$L_1\left[A_2 + B_2\left(\frac{-A_3}{B_3}\right)\right] + L_2\left[A_1 + B_1\left(\frac{-A_3}{B_3}\right)\right] = 0;$$

or

$$L_1(A_2B_3 - A_3B_2) + L_2(A_1B_3 - A_3B_1) = 0.$$

In determinant form this equation becomes:

$$\begin{vmatrix} L_1 & -L_2 & 0 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = 0.$$

It may be readily verified that the last equation represents the median upon the side $L_3=0$ even when $B_3=0$.

A COMMENT ON SNYDER'S CORRECTION FORMULA*

EVELYN FRANK, University of Illinois

The correction formula (2) of [5] goes back to Lagrange [4]. It is also found, for example, in [6], pp. 222–223, and has recently been repeated a number of times, for example, in [1] and [2]. In [3] it is extended to n equations in n unknowns.

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3. E. Frank, On the calculation of the roots of equations, Jrn. Math. Phys. vol. 34, 1955, pp. 187–197.

4. J. L. Lagrange, Traité de la Résolution des Équations Numériques de tous les Degrés, Oeuvres, vol. 8, Gauthier-Villars, Paris, 1879, Note XI, pp. 258–285.

5. Ralph W. Snyder, One more correction formula, this MONTHLY, vol. 62, 1955, pp. 722–725.

6. Fr. A. Willers, Practical Analysis, Dover Publications, Inc., New York, 1948.

* See [5].

ON THE APPLICATIONS OF THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

PAUL SCHILLO, University of Buffalo

In the usual elementary calculus course, it is an exceptional student who can avoid the misconception that the definite integral is the sum of infinitely many infinitesimal differentials. This is because, in most texts, none of the problems requiring the application of the Fundamental Theorem of the Integral Calculus discourage this notion. In fact, the integrals, which yield the correct solutions to these problems, are most easily set up by using this misconception as a device. For example, the area of the region bounded by the line $y=x$ and by the parabola $y=x^2$ can be thought of as the sum of the areas of all infinitesimal $x-x^2$ by dx rectangles from $x=0$ to $x=1$ and, consequently, can be written as:

$$\int_0^1 (x - x^2)dx = \tfrac{1}{6}.$$

Some texts attempt to remedy this situation by including among their exercises questions which ask for the values of limits like:

(1) $$\lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 \quad \text{and}$$

(2) $$\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin \frac{k\pi}{n}.$$

Of all those who have, at some time, been introduced to the integral calculus, few would feel competent to evaluate these limits. Some might be successful after much effort in showing the validity of the following not very general formulas:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \quad \text{and} \quad \sum_{k=1}^{n} \sin \frac{k\pi}{n} = \cot \frac{\pi}{2n}.$$

Then, applying certain theorems on limits, they could show that

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \frac{1}{4}$$

and

$$\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin \frac{k\pi}{n} = 2\left[\lim_{n\to\infty} \cos \frac{\pi}{2n}\right]\left[\lim_{n\to\infty} \frac{\frac{\pi}{2n}}{\sin \frac{\pi}{2n}}\right] = 2.$$

However, the student who has not been weaned from the correct definition

of the definite integral as the limit of a sequence of sums should be able to recognize (1) and (2) as the respective integrals:

$$\int_0^1 x^3 dx = \frac{1}{4} \quad \text{and} \quad \int_0^\pi \sin x dx = 2.$$

Thus, these limit problems help the student to learn just what the definite integral is.

Now, according to a widely held belief, the methods used in solving the so-called "practical" problems (*e.g.*, finding the area between curves *etc.*) are retained by students much longer than those used in solving straightforward drill problems of the same complexity. If this is true, much of the value of the limit problems is likely to be lost unless they are clothed in "practicality." For this reason, an elementary calculus course might advantageously include problems of the "ultimate-average" type.

Let S be a structure or situation, which can involve any arbitrary number of members of a certain kind; let A_n be some arithmetic average associated with S when it involves exactly n members; then, if $\operatorname{Lim}_{n\to\infty} A_n$ exists, this limit is called an ultimate average in S. For example, the ultimate average distance between pairs of vertices in a regular polygon circumscribed by a unit circle is $4/\pi$ units, because

$$A_n = \frac{2}{n-1} \sum_{k=1}^{n-1} \sin \frac{k\pi}{n}$$

and

$$\operatorname*{Lim}_{n\to\infty} A_n = \frac{2}{\pi}\left[\operatorname*{Lim}_{n\to\infty} \frac{n}{n-1}\right]\left[\operatorname*{Lim}_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin \frac{k\pi}{n}\right] = \frac{4}{\pi}.$$

Also, the ultimate average distance between pairs of points in a one unit column of equi-spaced points is $\frac{1}{3}$ units, because

$$A_n = \frac{2}{n^2(n+1)} \sum_{k=1}^{n} (n+1-k)k$$

and

$$\operatorname*{Lim}_{n\to\infty} A_n = 2\left\{\left[\operatorname*{Lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n}\right] - \left[\operatorname*{Lim}_{n\to\infty} \frac{n}{n+1}\right]\left[\operatorname*{Lim}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2\right]\right\}$$

$$= 2\left\{\int_0^1 x dx - \int_0^1 x^3 dx\right\} = \frac{1}{3}.$$

Since ultimate average problems are quite easy to compose, an instructor should have no trouble in giving his class a sufficient number of them.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1216. *Proposed by N. A. Court, University of Oklahoma*

The area of the triangle formed by the midpoints of three (not necessarily concurrent) cevians drawn through the three vertices of a given triangle is equal to one fourth of the area of the triangle determined by the feet of those cevians.

E 1217. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Evaluate

$$\prod_{d|n} d^{\phi(n/d)+\phi(d)}.$$

E 1218. *Proposed by Robert Spira, Berkeley, Calif.*

Consider the two propositions:

I. If $(a, b)=1$, then $ax+b$ assumes infinitely many prime values.

II. If $(a, b)=1$, then $ax+b$ assumes at least one prime value.

I is Dirichlet's theorem. Clearly I implies II. Show that II implies I.

E 1219. *Proposed by D. A. Robinson, University of Wisconsin*

The sequence $\{s_n\}$, where $s_1=\sqrt{2}$ and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$, has a finite limit. Find this limit in closed form.

E 1220. *Proposed by Judith Blankfield, University of Illinois*

It is well known that if

$$r_1 > r_2 > r_3 > \cdots \to 0,$$

then

$$\left| \sum_{t=n+1}^{\infty} (-1)^t r_t \right| \leqq r_n.$$

Prove the following generalization which reduces to the above case when $\theta=\pi$.

$$\left| \sum_{t=n+1}^{\infty} r_t e^{it\theta} \right| \leqq r_n/\sin(\theta/2), \qquad 0 < \theta < 2\pi.$$

SOLUTIONS

Total Number of Permutations of n Things

E 1186 [1955, 655]. *Proposed by David Freedman, Student, McGill University*

Show that the total number of permutations of n things is $[n!e]$, where $[x]$ denotes the greatest integer in x.

Solution by P. B. Johnson, Occidental College. The total number P of permutations of n things is the sum of the number of permutations taken n, $n-1$, $n-2$, $\cdots$, 1, 0 at a time. Hence

$$P = \sum_{r=0}^{n} n!/r! = n!e - \sum_{r=n+1}^{\infty} n!/r!.$$

But

$$0 < \sum_{r=n+1}^{\infty} n!/r! < \sum_{s=1}^{\infty} 1/(n+1)^s = 1/n \leqq 1,$$

whence the result.

Also solved by A. N. Aheart, Ferrel Atkins, S. K. Berberian, D. G. Brennan, G. B. Charlesworth, John Christopher, A. E. Danese, R. E. Daniels and J. D. E. Konhauser (jointly), Marcel Delcourte, J. M. Elkin, Michael Goldberg, D. S. Greenstein, Nathaniel Grossman, M. J. Hellman, Vern Hoggatt, P. W. M. John, E. S. Keeping, D. C. B. Marsh, F. E. Milne, T. F. Mulcrone, C. S. Ogilvy, Hiram Paley, Walter Penney, C. F. Pinzka, L. A. Ringenberg, Azriel Rosenfeld, C. M. Sandwick, Sr., J. P. Scholz, W. A. Small, Bernard Smilowitz, O. E. Stanaitis, Art Steger, Chih-yi Wang, Alan Wayne, David Zeitlin, and the proposer. Late solutions by A. C. Goddard, A. R. Hyde, F. I. John, and M. S. Klamkin.

A Limit Problem

E 1187 [1955, 655]. *Proposed by James Goode, Student, Georgia Institute of Technology*

Find

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k.$$

I. *Solution by G. B. Charlesworth, Hofstra College.* By areas

$$\int_{n}^{2n+1} dx/x < \sum_{k=n}^{2n} 1/k < \int_{n-1}^{2n} dx/x,$$

whence

$$\ln (2n+1)/n < \sum_{k=n}^{2n} 1/k < \ln 2n/(n-1).$$

Therefore

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k = \ln 2.$$

II. *Solution by A. E. Danese, Mount Morris, N. Y.* We have

$$\sum_{k=n}^{2n} 1/k = \sum_{k=1}^{2n} 1/k - \sum_{k=1}^{n-1} 1/k$$

$$= \left(\sum_{k=1}^{2n} 1/k - \ln 2n \right) - \left(\sum_{k=1}^{n} 1/k - \ln n \right) + 1/n + \ln 2.$$

Since the limit of each parenthesis is Euler's constant γ, we have

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k = \ln 2.$$

III. *Solution by O. E. Stanaitis, St. Olaf College.* From the definition of a definite integral

$$\lim_{n\to\infty} \sum_{k=n}^{2n} 1/k = \lim_{n\to\infty} (1/n) \sum_{k=0}^{n} 1/(1 + k/n) = \int_0^1 dx/(1 + x) = \ln 2.$$

IV. *Solution by Albert Wilansky, Lehigh University.* One may easily show by induction that

$$\sum_{k=n+1}^{2n} 1/k = \sum_{k=1}^{2n} (-1)^{k-1}/k,$$

whence the required limit is ln 2.

Also solved by D. S. Adorno, W. A. Al-Salam, J. E. Darraugh, Marcel Delcourte, Hüseyin Demir, I. A. Dodes, E. S. Eby, J. M. Elkin, Calvin Foreman, A. M. Glicksman, Michael Goldberg, D. S. Greenstein, R. E. Greenwood, Cornelius Groenewoud, Nathaniel Grossman, M. J. Hellman, A. S. Hendler, T. W. Hildebrandt, Vern Hoggatt, W. H. Hooker, F. A. Homann, P. W. M. John, P. B. Johnson, L. M. Jordan, E. S. Keeping, M. A. Kirchberg, J. D. E. Konhauser, H. D. Lipsich, Kenneth Loewen, D. C. B. Marsh, Eugene McLachlan, Elliott Mendelson, J. D. Miller, T. A. Newton, C. S. Ogilvy, Ingram Olkin, Hiram Paley, F. D. Parker, Walter Penney, W. J. Pervin, L. A. Ringenberg, Azriel Rosenfeld, J. W. Ross, C. M. Sandwick, Sr., L. P. Scholz, D. D. Strebe, Chih-yi Wang, R. M. Warten, Alan Wayne, David Zeitlin, and the proposer. Late solutions by A. R. Hyde, F. I. John, M. S. Klamkin, and C. F. Pinzka.

An Orthocentric Tetrahedron

E 1189 [1955, 655]. *Proposed by N. A. Court, University of Oklahoma*

If two pairs of spheres with non-coplanar centers are such that each sphere

of one pair is orthogonal to the two spheres of the other pair, then the tetrahedron formed by the centers of similitude of the two pairs of spheres is orthocentric.

Solution by P. D. Thomas, Air Force Armament Center, Eglin Air Force Base, Florida

A consequence of the orthogonality conditions stated is that the two coaxial pencils determined by the two pairs of spheres as given are orthogonal. That is, every sphere of either pencil is orthogonal to every sphere of the other, and of course the two lines of centers are orthogonal.

Now the sphere of similitude of each given pair (the sphere having for diameter the segment determined by the centers of similitude of the pair) belongs to the coaxial pencil determined by that pair, and the two respective spheres of similitude are orthogonal.

But the ends of two skew rectangular diameters of two orthogonal spheres are vertices of an orthocentric tetrahedron. Hence the proposition.

See N. A. Court, *Modern Pure Solid Geometry*, pp. 153, 168, 179, 185, 189, 269.

Also solved by Roscoe Woods and the proposer.

A Lucky Larry

E 1190 [1955, 655]. *Proposed by R. A. Miller, University of Mississippi*

A student wrote the following in solving the problem, "Find the value of $(3+x)^{2/3}$ when $x=3/8$:"

$$(3 + x)^{2/3} = (3 + 3/8)^{2/3} = 2 + 1/4 = 9/4.$$

Investigate the conditions under which

$$(a + b/c)^{m/n} = (a + b/c)(m/n),$$

where a, b, c, m, n are positive integers.

Solution by E. P. Starke, Rutgers University. First we note that two different positive integral k ($k>1$) powers must differ by more than k, since

$$r^k - s^k = (r - s)(r^{k-1} + r^{k-2}s + \cdots + s^{k-1}),$$

and $r-s \geqq 1$ and the second expression in parentheses exceeds k.

Consider

$$u^{m/n} = u(m/n),$$

where u is positive and rational, and m, n are relatively prime positive integers. If $m>n$, we have $u^{m-n}=(m/n)^n$. Since $m-n$ and n are relatively prime, both m and n must be perfect $(m-n)$th powers. But they differ by $m-n$. By our earlier remark this is impossible unless $m-n=1$. Hence $m=n+1$ and

$$u = a + b/c = (1 + 1/n)^n$$

with n arbitrary.

The case $m=n$ is trivial. If $m<n$, we have $(n/m)^n=u^{n-m}$ and, by the same argument, $n=m+1$ and

$$u = a + b/c = (1 + 1/m)^{m+1}.$$

The case cited by the Proposer results from $n=3$, $m=2$.

Also solved by G. B. Charlesworth, Marcel Delcourte, I. A. Dodes, Michael Goldberg, Cornelius Groenewoud, Virginia Hanly, D. C. B. Marsh, O. C. Nussmann, C. S. Ogilvy, Walter Penney, Susan Pyeatt, Azriel Rosenfeld, G. J. Simmons, O. E. Stanaitis, Chih-yi Wang, R. M. Warten, H. R. Wesson, and Hazel Schoonmaker Wilson. Late solutions by A. R. Hyde, F. I. John, M. S. Klamkin, and M. A. Rashid.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4688. *Proposed by A. H. Clifford, Tulane University*

What positive integers n have the property that every positive integer less than n is expressible as the sum of distinct divisors of n?

4689. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $f(x)$ be any function such that $f'''(x)\geqq 0$, $f(n)\sim f(n+1)$. Prove that

$$\sum_{n=0}^{\infty}(-1)^n t^{f(n)}$$

tends to $\frac{1}{2}$ as $t\to 1^-$.

4690. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Being given a tetrahedron $ABCD$ and the tetrahedron $A_1B_1C_1D_1$ obtained by passing planes through A, B, C, D parallel to the opposite faces of $ABCD$, show that

$$PA^2 + PB^2 + PC^2 - 2PD^2 - PD_1^2$$

is a constant independent of the position of point P. Extend this property to a skew polygon of n vertices.

4691. *Proposed by R. E. Johnson, Smith College*

A weakly prime ideal of a ring R is any ideal I having the property that either $aR \subset I$ or $Ra \subset I$ implies that a is in I. Give an elementwise characterization of the unique minimal weakly prime ideal of R.

4692. *Proposed by A. E. Currier, United States Naval Academy*

Given $f(x)$ defined by

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n x^{2^n},$$

prove that

$$(1) \qquad -\frac{3+8\sqrt{6}}{750} < \liminf_{x \to 1-} f(x) < -\frac{1}{750},$$

$$(2) \qquad \frac{1}{750} < \limsup_{x \to 1-} f(x) < \frac{3+8\sqrt{6}}{750}.$$

SOLUTIONS

Digits of the Fibonacci Numbers

4609 [1954, 646]. *Proposed by Vern Hoggatt, San Jose State College, California*

Let $q_k(F_n)$ be the kth digit (from the right) of F_n, the nth Fibonacci number ($F_0=0$, $F_1=1$, $F_{n+2}=F_{n+1}+F_n$). Let s be defined such that $q_k(F_n)=0$ for $n=0, 1, 2, \cdots, s$ but $q_k(F_{s+1}) \neq 0$. (1) Show that the sequence $q_k(F_n)$ has at least another run of $s+1$ consecutive zeros. (2) Prove or disprove: the sequence $q_k(F_n)$ has never more than $s+1$ consecutive zeros.

Editorial Note. No communication regarding this problem has been sent in, but some closely related questions are answered in the Proposer's paper, A type of periodicity for Fibonacci Numbers, *Mathematics Magazine*, vol. 28, 1955, pp. 139–142.

Monic Irreducible Polynomials

4624 [1955, 126]. *Proposed by H. S. Shapiro, New York University*

Let J_p denote the field of integers modulo p, where p is a prime, and $J_p[x]$ the ring of polynomials with coefficients in J_p. Then the number of monic irreducible polynomials of degree n in $J_p[x]$ is equal to

$$\frac{1}{n} \sum_{d|n} \mu(d) p^{n/d}$$

where μ is the Möbius function.

Solution by B. M. Stewart, Michigan State University. A polynomial of degree s which is irreducible in $J_p(x)$ divides $x^{p^n}-x$ if and only if s divides n. (Ex. 1, p. 180, MacDuffee, *Introduction to Abstract Algebra.*) Furthermore, $x^{p^n}-x$ has no repeated factors. Hence if $N(p, s)$ indicates the number of distinct monic irreducible polynomials of degree s in $J_p(x)$, then

$$p^n = \sum sN(p, s)$$

summed over the positive divisors s of n.

This is a perfect situation for applying the inversion formula which involves the Möbius function μ. The result is

$$nN(p, n) = \sum \mu(d)p^{n/d}$$

summed over the positive divisors d of n.

Also solved by P. T. Bateman, Robert Breusch, L. Carlitz, L. E. Clarke, Harley Flanders, S. W. Golomb, D. C. B. Marsh, Arnold Walfisz, Koichi Yamamoto, and the Proposer.

Editorial Note. The result may be found in many places including Dedekind, *Journ. reine u. angew. Math.*, vol. 54 (1857), pp. 19–22; L. E. Dickson, *Linear Groups* (1901), p. 18; H. Hasse, *Zahlentheorie* (1949) p. 32—references supplied by readers.

Legendre Polynomials

4628 [1955, 186]. *Proposed by L. Carlitz, Duke University*

Let p be an odd prime and let $P_m(x)$ denote the Legendre polynomial of degree m, where $p=2m+1$. Show that

$$P_m(3) \equiv P_m(-3) \equiv \begin{cases} 0 \pmod p & (m \text{ odd}) \\ 2a \pmod p & (m \text{ even}), \end{cases}$$

where a is the unique odd integer determined by

$$p = a^2 + b^2, \qquad a \equiv b + 1 \pmod 4.$$

Solution by the Proposer. Since $P_m(-x)=(-1)^m P_m(x)$, it suffices to discuss $P_m(3)$.

We shall make use of the following well known formula for the Legendre polynomial:

$$P_m(1 - 2x) = F(-m, m + 1; 1; x), \tag{1}$$

where $F(a, b; c; x)$ denotes the hypergeometric function. (For proof, see Whittaker and Watson, *Modern Analysis*, 4th edition, p. 311).

Now it is clear from (1) that

$$P_m(1-2x) \equiv \sum_{k=0}^{m} \binom{m}{k}^2 x^k \pmod{p}.$$

For $x=-1$ this becomes

$$P_m(3) \equiv \sum_{k=0}^{m} (-1)^k \binom{m}{k}^2 \pmod{p}. \tag{2}$$

For m odd the sum in the right member of (2) evidently vanishes since the kth term and the $(m-k)$th term cancel. Hence $P_m(3) \equiv 0 \pmod{p}$ in this case.

When m is even, let $m=2n$; then we have

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}. \tag{3}$$

This may be proved by comparing coefficients of x^{2n} in both members of the identity

$$(x-1)^{2n}(x+1)^{2n} = (x^2-1)^{2n}.$$

It follows from (2) and (3) that

$$P_{2n}(3) \equiv (-1)^n \binom{2n}{n} \pmod{p=4n+1}.$$

But by a formula of Gauss (Theorie residuorum biquadraticorum, *Werke*, vol. 2, p. 90)

$$\frac{1}{2^{2n}} \binom{2n}{n} \equiv 2a \pmod{p},$$

where a is the unique odd integer determined by $p=a^2+b^2$, $a \equiv b+1 \pmod 4$. Since $2^{2n} \equiv (2/p) = (-1)^{(p^2-1)/8} = (-1)^n$, where $(2/p)$ is the Legendre symbol, it follows that $P_{2n}(3) \equiv 2a \pmod{p}$.

The following additional remark has some interest. If $p=2m+1 \equiv 3 \pmod 4$, then

$$\frac{1}{p} P_m(3) \equiv -\sum_{r=1}^{m} (-1)^r \binom{m}{r}^2 \left(1 + \frac{1}{2} + \cdots \frac{1}{r}\right) \pmod{p}.$$

The problem was suggested by its connection with elliptic functions and particularly with the lemniscate function (see A. Hurwitz, *Mathematische Annalen*, vol. 51 (1899), pp. 196–226; *Mathematische Werke*, Basel, 1933, vol. 2, pp. 342–373). The following general result may be stated. Assume that the elliptic function sn x admits of complex multiplication so that the period quotient belongs to an imaginary quadratic field of discriminant d. Let k^2 denote the corresponding singular modulus and let p be an odd prime such that the Legendre symbol $(d/p) = -1$. Then $P_m(1-2k^2) \equiv 0 \pmod{p}$. See *Mathematische*

Annalen, vol. 127 (1954), pp. 162–169 and *Monatshefte für Mathematik*, vol. 58 (1954), pp. 77–90. This result does not, however, determine the residue of $P_m(1-2k^2) \pmod{p}$ when $(d/p)=+1$.

Subring of a Radical Ring

4629 [1955, 186]. *Proposed by M. Henriksen and S. Perlis, Purdue University*

A commutative ring R is called a radical ring (in the sense of Jacobsen) if every element has a quasi-inverse; *i.e.*, for every a in R, there is an x in R such that $a+x+ax=0$. In particular, if every element of R is nilpotent, then R is a radical ring. Need every subring of a radical ring be a radical ring?

I. *Solution by A. J. Goldman, Princeton University.* We show first that an arbitrary commutative ring R (radical or not) without unity can be imbedded in a radical ring R''. Imbed R in a commutative ring R' with unity (it is well known that this can be done) and let F be the quotient field of R'. Then R is imbedded in the subring R'' of F consisting of all elements of the form $a/(1+b)$, where a and b range through R; it is clear that R'' is actually a ring. R'' is radical because $a/(1+b)$ has the quasi-inverse $-a/(1+a+b)$.

To answer the question in the negative, it now suffices to exhibit a commutative ring R without unity which is not radical. The even integers form such a ring, since the element 2 has no quasi-inverse.

II. *Solution by P. M. Cohn, the University, Manchester, England.* Let R be the ring of formal power series in x over a field F, with zero constant term. Then R is a radical ring since any f in R has the quasi-inverse $-f+f^2-f^3+\cdots$. The polynomials in R form a subring S which is not a radical ring since it contains x but no quasi-inverse of x. For if

$$f = a_1x^n + \cdots + a_nx \qquad (a_1 \neq 0)$$

were a polynomial satisfying $x+f+xf=0$, then, by equating the coefficient of x^{n+1} to zero we obtain $a_1=0$, which is a contradiction. Thus no non-zero element of S is a quasi-inverse of x; also zero is not.

Also solved by W. R. Abel and C. D. Gorman, E. H. Batho, S. W. Golomb, C. R. Phelps, Gustave Rabson, R. B. Reisel, D. W. Sasser, W. R. Scott, M. F. Smiley, M. L. Tomber, Jerrold Yos, and the Proposer.

A Continued Fraction Related to the Bessel's Functions

4631 [1955, 187]. *Proposed by F. H. Northover, Memorial University of Newfoundland*

Evaluate the unending continued fraction

$$1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\cdots.$$

Solution by D. J. Newman, Republic Aviation Corporation, Farmingdale,

N. Y. Applying the method of continued fractions to the differential equation $y=xy''$ (See Ince, *Differential Equations*, p. 178) we obtain

$$x\frac{y'}{y} = 1 + \frac{x}{2+}\,\frac{x}{3+}\,\frac{x}{4+}\cdots$$

for some solution y. (Namely, letting $x\to 0$, it is that solution for which $y(0)=0$, $y'(0)=1$). All that remains then is to solve the equation

$$y = xy'', \qquad y(0) = 0, \qquad y'(0) = 1$$

and our answer will be $y'(1)/y(1)$.

The differential equation is of the Bessel type and the required solution is then $y=x^{1/2}I_1(2x^{1/2})$, whence $y'=I_0(2x^{1/2})$. This gives as our final answer $I_0(2)/I_1(2)$, or in series form

$$\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\Big/\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}.$$

A numerical value, 1.433127, is easy to obtain either from the series or from convergents to the continued fraction.

Also solved by Leonard Carlitz, M. Delcourte, V. C. Harris, P. G. Kirmser, D. H. Lehmer, Erich Michalup, C. D. Olds, C. F. Pinzka, C. C. Sams, R. E. Shafer, M. R. Spiegel, Chih-yi Wang, Alan Wayne, and the Proposer.

Editorial Note. The proposed continued fraction is a special case of many in the literature. Solvers referred to Chrystal, *Algebra*, vol. 2, p. 494, and pp. 520–522; Erdelyi, *et al.*, *Higher Transcendental Functions*, vol. 2, formula 7.11(23); H. S. Wall, *Continued Fractions*, p. 349; Perron, *Die Lehre von der Kettenbrüchen*, p. 299 and pp. 479–480.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Transform Calculus with an Introduction to Complex Variables by E. J. Scott. Harper and Brothers, New York, 1955. 8+330 pages. $7.50.

In the introduction the author indicates that this book is directed toward advanced undergraduate and graduate students, mainly in physics, chemistry, and engineering. For this audience it is excellent, providing them with many techniques and specific results of much importance in their work. Mathematical rigor is attempted within reason, but the author's emphasis is more on vigor than on rigor. He does not attempt the most general known statement of a

theorem if its proof would take him through circuitous pedantics. Thus he confines himself to piecewise continuity in his statement of the Fourier integral theorem (which he does not prove). Some theorems are stated imperfectly, and the English is not always perfect. Some discussions might be written more satisfactorily from the logician's viewpoint, but it would be difficult to do this without spoiling their pedagogic value. There are some misprints.

Chapter I gives the highlights of the theory of functions of a complex variable. Chapter II introduces the Fourier and Laplace transforms and their inverses. Some highlights of the theory of the Laplace transform are given in Chapter III, and are applied to ordinary differential equations with constant coefficients in Chapter IV. Chapter IV contains a number of examples of practical interest from electric circuit theory and elasticity.

In Chapter V matrix concepts are introduced and applied to systems of linear differential equations of the type that occur in multiple degree of freedom electrical and mechanical systems. These are solved using the Laplace transform. A number of examples involving actual applications to electrical and mechanical networks are given.

In Chapter VI, the Laplace transform is applied to the solution of difference equations; in particular several applications are made to mechanical and electrical problems.

Chapter VII is a long chapter in which the Laplace transform is applied to the solution of linear partial differential equations. Diffusion is treated in detail, and some transmission line problems are solved. Longitudinal and transverse vibrations of a bar are treated, as well as transverse vibrations in a string. The emphasis is upon the particular applications. Nothing like a unified theory of elliptic, hyperbolic, or parabolic equations is attempted.

In Chapter VIII, Gamma, Error, Fresnel, and Bessel functions are briefly introduced, as well as asymptotic series and some simple integral equations. The author is to be commended for employing the standard notation $Y_n(t)$ for the now universally employed second solution of the Bessel equation which is due to Weber. Jahnke and Emde erroneously attributed this function to Neumann and changed the notation to $N_n(t)$. Unfortunately many electrical engineers in this country have followed this schismatic practice.

Chapter IX treats some partial differential equations of a more complicated nature which involve the special functions introduced in Chapter VIII. A number of problems of practical interest are worked.

In the last chapter, Chapter X, the author introduces what he calls finite Fourier transforms. The resulting theory is equivalent to the theory of Fourier series. He also introduces what he calls the finite Hankel transform. The resulting theory is equivalent to the theory of Fourier-Bessel series.

This book will provide engineers and physical scientists with a good many techniques that they will find important. It will not be satisfactory as a last word on the subjects treated to the mathematics major (for instance, a mathematics major ought to be able to *prove* Fourier's theorem). On the other hand,

a good many mathematics students would be much better off for having had a course of this type. There might then be fewer members of that forlorn group of non-productive Ph.D.'s who know theorems precisely, but have no idea what in the world to *do* with them.

A valuable feature of the book is its many examples and illustrations. The typography is excellent.

E. PINNEY
University of California
Berkeley, California

Numerical Methods. By A. D. Booth. New York, Academic Press. 1955. 7+195 pages. $6.00.

This is a small book (187 pages) covering a wide range of topics in Numerical Analysis. Since it covers such a wide range of topics the treatment of each subject is necessarily brief. The author states that the book was prepared with automatic computing equipment in mind; therefore, there is emphasis on methods suitable for use with automatic computing equipment.

The book starts with interpolation formulas using finite differences. Numerical integration and differentiation is the subject of the next chapter. Summation of series and the solution of ordinary and partial differential equations, as well as the solution of linear and non-linear systems of equations, are treated. The book ends with approximating functions, Fourier synthesis and analysis, and Monte Carlo methods.

There are some misprints in the book. For example, on Page 2 read SWAC for SEAC. Clearly, formula 4.6.12 on Page 39 should read $s_m y^m/m$ instead of $s_m y^m$. In many places in Chapter 6 the error terms of formulas involve factors such as $f^{(6)}$ max. These should be replaced by terms of the form $f^{(6)}(s)$ where s is some point in the interval under consideration. With these indications of lack of care in the statement of formulas, one realizes that other formulas in the book must be used with caution.

However, the development of high speed computing equipment has increased the demand for material on numerical analysis so this book should be part of the library of all those interested in computation.

H. D. HUSKEY
University of California

Automatic Feedback Control System Synthesis. By J. G. Truxal. McGraw-Hill, New York. Electrical and Electronic Engineering Series, 1955, xiii+675 pages. $12.50.

This book is intended as an engineering text and reference. It achieves the second aim much more successfully than the first. The book is based on a second course in servomechanisms as taught by the author. The book contains many cases where initial references are made apparently to content material of the

first course. In some of these instances a text reference is given, but this procedure is confusing when reading the book *per se*.

The book is replete with completely solved examples. In fact there is a tendency to explain by using many specific illustrations. The book might have been considerably reduced in size with little loss if proofs were given or generalized illustrations used. The detail with which examples are worked out is also excessive. At this level for example, it should not be necessary to work out polynomial long divisions in detail.

The contents are divided as follows:

Chapter 1, Introduction; develops Laplace Transform Techniques.

Chapter 2, Signal-flow Diagrams and Feedback Theory; and Chapter 3, Synthesis of RC Networks; develop the basic theory of network synthesis and circuit stability.

Chapter 4, Root-locus Methods; Chapter 5, Synthesis through Pole-zero Configurations; and Chapter 6, Design in the s Plane; constitute the major contribution of the book. Here the details of techniques for linear network synthesis are described fully.

Chapter 7, Principles of Statistical Design; Chapter 8, Application of Statistical Design Principles; and Chapter 9, Sample-data Control Systems; develop the statistical treatment leading to system design techniques where noise is involved.

Chapter 10, Non-linear Systems and Describing-function Analysis; and Chapter 11, Phase-plane Analysis; discuss several methods of treating non-linear systems.

From a mathematical point of view it is distressing to find the author accentuating Laplace Transform theory as the only essential mathematical tool necessary for servo-engineers. A much healthier approach would be to accentuate the value of a strong mathematical background for all engineers.

Generally the book is well written and well printed. Few misprints were noted. The book is comprehensive and should prove useful as a reference for the field.

PAUL BROCK
Electro Data Corporation
Pasadena, California

NEW BOOKS RECEIVED

Modern Trigonometry. By W. A. Turledge and J. A. Pond. Englewood Cliffs, N. J., 1956. 11+243 pages.

An Introduction to Mathematics. By L. E. Boyer. New York, Henry Holt and Co., 1956. 16+528 pages. $5.25.

Nellie Landblom's Copybook for Beginners in Research Work. By Nellie Thompson Landblom. Multigraphed book available from author, P. O. Box 248, Fort Collins, Colorado, 1955. 118 pages. $2.95.

Elements of Business Mathematics for Colleges. Llewellyn R. Snyder. New York, McGraw-Hill Book Co., 1956. x+249 pages. $3.75.

Carl Friedrich Gauss: Titan of Science. A Study of His Life and Work. By G. W. Dunnington. New York, Exposition Press, 386 Fourth Ave., New York 16, N. Y., 1955. xi+479 pages. $6.00.

Surface Area. By Lamberto Cesari. Princeton, N. J., Princeton University Press, 1956. x+595 pages. $8.50.

Articles et Memoires—I, La variable complexe, II, Le champ reel—Notices—Arnaud Denjoy. Paris, Gauthier-Villars, 1955. Vol. 1, x+507 pages and Vol. 2, vi+600 pages. 2 volumes $15.03.

Calculus, Differential and Integral, with Problems and Solutions. By G. M. Peterson and R. F. Graesser. Ames, Iowa. Littlefield, Adams and Co., 1956. 10+321 pages. $1.75.

Through the Mathescope. By C. S. Ogilvy. New York, Oxford University Press. 1956. viii+162 pages. $4.00.

Elementary Topology. By D. W. Hall and G. L. Spencer. New York, John Wiley and Sons, 1955. xii+303 pages.

Tracts for Computers, No. XXVI, Correlated Random Normal Deviates. By E. C. Fieller, T. Lewis and E. S. Pearson. New York, Cambridge University Press, 1956. 60 pages. $2.00.

Spheroidal Wave Functions, Including Tables of Separation Constants and Coefficients. By J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little and F. J. Corbato. Published jointly by the Technology Press of M.I.T. and John Wiley and Sons, Inc., New York, 1956. xiii+613 pages, $12.50.

Plane Waves and Spherical Means Applied to Partial Differential Equations. By Fritz John. New York, Interscience Publishers, Inc., 1955. 180 pages. $4.50.

Experimental Design. By W. T. Federer. New York, The Macmillan Company, 1955. xix+544+47 pages of problems. $11.00.

Mathematics of Finance, Second Edition. By P. M. Hummel and C. L. Seebeck, Jr. New York, McGraw-Hill Book Company, 1956. xi: 372 pages. $4.75.

Analytic Geometry, Fifth Edition. By C. E. Love and E. D. Rainville. New York, The Macmillan Co., 1955. xiv+302 pages. $4.00.

Elementary Statistics for Students of Social Science and Business. By R. C. Sprowls. New York, McGraw-Hill Book Company, 1955. xiii+392 pages. $5.50.

Analytic Geometry and Calculus. By T. S. Peterson. New York, Harper and Brothers, 1956. ix+456 pages. $5.50.

Introduction to Numerical Analysis (International Series in Pure and Applied Mathematics). By F. B. Hildebrand. New York, McGraw-Hill Book Company, 1956. x+511 pages. $8.50.

Mathematical Theory of Elasticity, Second Edition. By I. S. Sokolnikoff. New York, McGraw-Hill Book Company, 1956. xi+476 pages. $9.50.

Plane Trigonometry, Third Edition. By A. L. Nelson and K. W. Folley. New York, Harper and Brothers, 1956. xi+134 pages. $3.50.

Numerical Analysis. By Zdenek Kopal. New York, John Wiley and Sons, Inc., 1955. xiv+556 pages. $12.00.

Methods in Numerical Analysis. By K. L. Nielsen. New York, The Macmillan Company, 1956. viii+382 pages. $6.90.

An Introduction to Linear Algebra. By L. Mirsky. New York, Oxford University Press, 1956. xii+433 pages. $5.60.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1956.

Michigan State University. June 26 to August 3: Professor Arnold, correlation analysis; Professor Hill, advanced calculus I, functions of a complex variable; Professor J. B. Kelly, differential equations, matrices and groups; Professor Meixner, potential theory, mathematical theory of continuous matter; Professor Stewart, projective geometry. June 26 to August 24: Dr. Berberian, mathematics of engineering; Professor Blair, functions of a real variable, algebraic systems of continuous functions; Dr. Faith, advanced calculus II; Professor Hannan, analysis of variance; Professor L. Kelly, differential equations, vector analysis, combinatorial topology; Mr. Laurent, statistical methods in engineering.

Stanford University. June 25 to August 17: Professor Weinberger (University of Maryland), approximation of quadratic functionals; Professor Lax (New York University), selected topics from analysis; Professor Spencer (Princeton University), seminar.

State University of Iowa. June 12 to August 8: Professor Hogg, introduction to mathematical statistics, statistical hypotheses; Professor Johnson, astronomy; Professor Knowler, operations research, actuarial aspects of casualty insurance; Professor Muhly, advanced calculus, group theory; Professor Price, the supervision of mathematics, laboratory in the construction of teaching aids in mathematics; Professor Smiley, matrices and determinants, introduction to advanced mathematics; Professor Woods, studies in secondary geometry, differential equations.

University of Michigan, School of Education. June 25 to July 13: The Workshop for College Professors will present a program designed to help college teachers develop new ideas and fresh materials for their academic courses. Such topics as course planning, teaching techniques, and evaluation will be discussed. For further information write to Director A. D. Henderson, 2442 U.E.S., University of Michigan, Ann Arbor, Michigan.

University of Minnesota, College of Science, Literature, and the Arts. June 11 to July 14: Professor Carlson, differential equations, non-euclidean geometry; Dr. Orey, Fourier series and orthogonal functions; Visiting Professor Roy, analysis of variance and multivariate analysis II, statistical models in the social sciences. July 16 to August 18: Professor Gelbaum, integral equations, intermediate differential equations; Professor Hatfield, Jr., Fourier series and orthogonal functions, II. June 18 to July 21: A Summer Institute for High School Teachers of Mathematics, carrying graduate credit, will be given with the support of the Hill Family Foundation.

University of Pittsburgh. June 11 to July 20, July 23 to August 31: Professors Blumberg and Christiano, differential equations, advanced calculus; Professor Bryson, vector analysis; Professor Taylor, functions of a complex variable; Professor Levine, modern algebraic theories; Professor Laush, infinite series, Lie theory of groups (second six weeks); Professor Bompiani, geometry of differential elements; Professor Elyash, advanced function theory (first six weeks). June 18 to August 10 (evenings): Professor Knipp, mathematics of modern engineering; Professor Bryson, Laplace transform theory and applications (second half of two semester course). July 2 to August 10: Professor Vogeley, differential equations for students of engineering; Professor Edwards, mathematical theory of statistics; Professor Teats, history of mathematics; Professor Myers, recreational mathematics for teachers, theory of equations; Mr. Kachun, navigation for teachers; Professor Knipp, teaching of secondary school mathematics.

University of Virginia. June 18 to August 14: Professor Ball, advanced calculus and applied mathematics; calculus of variations; Professor Floyd, topological groups. July 3 to August 14: Professor Ball, foundations of geometry.

CLIFFORD BREWSTER UPTON FELLOWSHIP

The American Book Company has established a fellowship in honor of Professor Emeritus Clifford Brewster Upton for graduate study in the Teaching of Mathematics at Teachers College, Columbia University. Three thousand dollars will be awarded annually to a student in the teaching of mathematics in elementary schools, in secondary schools, or in teacher training institutions. Inquiries should be made to the Chairman of the Committee on Fellowships and Scholarships, Teachers College, Columbia University, New York 27, New York, before May 1, 1956. Formal applications, together with all supporting materials, must be filed with the Committee by June 1, 1956.

CALIFORNIA CONFERENCE FOR TEACHERS OF MATHEMATICS

The 1956 California Conference for Teachers of Mathematics will be held at the University of California at Los Angeles from June 20 through July 3. The purpose of this conference is to bring together teachers interested in mathematics—arithmetic through calculus—to study common problems in the teaching of mathematics. For further information write to University of California Extension, Mathematics, Los Angeles 24, California.

PERSONAL ITEMS

Assistant Professor J. W. Lindsay of Texas Technological College has been awarded a Danforth Foundation Teachers Study Grant for the year 1956–1957. He will spend the year at the University of California at Los Angeles.

Illinois Institute of Technology announces the following: Mr. Michel McKiernan, previously a graduate assistant at the Institute, and Dr. Abe Sklar, formerly an assistant at California Institute of Technology, have been appointed to instructorships.

Kansas State College reports: Professor P. M. Young has been appointed Associate Dean of the School of Arts and Sciences; Professor R. G. Sanger, formerly the Acting Associate Dean of the School of Arts and Sciences, has resumed his duties as Head of the Department of Mathematics.

At Wartburg College: Professor J. O. Chellevold, head of the Department of Mathematics, has been appointed Acting Dean of the Faculty; Mr. Richard Wideranders of Estherville Junior College, Iowa, has been appointed to an instructorship.

Mr. R. G. Albert, formerly a graduate student at Harvard University, has a position as a programmer for the Datamatic Corporation, Newton Highlands, Massachusetts.

Mr. P. N. Armstrong, previously a technical engineer for the International Business Machines Corporation, Endicott, New York, is an engineer for the Magnavox Company, Santa Monica, California.

Mr. R. V. Benson of Los Angeles City College has been appointed to an instructorship at Long Beach City College.

Dr. W. J. Berger, formerly a mathematician at R.C.A. Service Company, Patrick Air Force Base, Cocoa, Florida, now has a position as a mathematician with the Raydist Navigation Corporation, Hampton, Virginia.

Assistant Professor D. R. Bey of Illinois State Normal University has been promoted to an associate professorship.

Dr. E. E. Blanche, previously an analyst with the Research and Development Division, Department of the Army, Washington, D. C., is President and Senior Research Scientist of Ernest E. Blanche and Associates, Rockville, Maryland.

Mr. H. L. Carlson, formerly an electronic applications specialist for the

National Cash Register Company, Dayton, Ohio, is employed as a computing engineer by Douglas Aircraft Company, Santa Monica, California.

Mr. D. A. Cope, recently an engineer for the Glenn L. Martin Company, Baltimore, Maryland, has accepted a position as a senior research engineer for the Consolidated-Vultee Aircraft Company, San Diego, California.

Mr. H. N. Garber, formerly National Science Foundation fellow at Massachusetts Institute of Technology, has a position as a management assistant with the Radio Corporation of America, Camden, New Jersey.

Miss Jane C. Ingersoll, previously an operations analyst at the Operations Research Office, Chevy Chase, Maryland, has been appointed to an instructorship at the University of Maryland.

Mr. Seymour Landau, formerly an instructor at the University of Connecticut, is now an actuary for the New York City Transit Authority, New York.

Mr. R. L. Mentzer of Teachers College of Connecticut has been promoted to an assistant professorship.

Associate Professor E. N. Nilson of Trinity College, Connecticut, has a position as a project engineer for Pratt and Whitney Aircraft Corporation, East Hartford, Connecticut.

Dean Emeritus and Professor J. R. Overman of Bowling Green State University is retiring in June 1956. He was the first member of the teaching staff, first Chairman of the Department of Mathematics, and first Dean of the College of Liberal Arts, Dean of Men, and Dean of Faculties. The University's Board of Trustees recently honored Dr. Overman by naming the new science building Overman Hall "in recognition of his unique service and contribution in the development and progress of the University."

Mr. J. L. Pallone, assistant actuary for Woodward and Fondiller, Inc., New York, New York, has been appointed to an instructorship at Polytechnic Institute of Brooklyn.

Assistant Professor H. C. Parrish of North Texas State College has been promoted to an associate professorship.

Mr. J. B. Roberts of Reed College has been promoted to an assistant professorship.

Dr. D. T. Sigley, formerly a physicist at Johns Hopkins University Physics Laboratory, is employed as Associate Director of the Engineering Laboratories, American Machine and Foundry Company, Greenwich, Connecticut.

Assistant Professor James C. Smith, Jr., of Loyola University of Los Angeles is a teacher in the Extension Division, University of California at Los Angeles.

Mr. G. E. Waldman, a teacher at Tottenville High School, Staten Island, New York, has been appointed Acting Chairman of the Department of Mathematics.

Dr. E. A. Walker, recently a graduate student at the University of Kansas, is employed as a mathematician at the National Security Agency, Washington, D. C.

Dr. G. P. Weeg, a mathematician at Engineering Research Associates, St. Paul, Minnesota, is now employed in the Systems Analysis Department, Remington Rand Univac Division of the Sperry Rand Corporation, St. Paul, Minnesota.

Mrs. Elizabeth S. Wolf, formerly an instructor at Indiana Technical College, has a position as a computer in the Advanced Development Division of the Avco Manufacturing Company, Stratford, Connecticut.

Dr. Albert Wolinsky, previously an associate physicist with the Farrand Optical Company, New York, New York, has a position as a staff member of the General Precision Laboratory, Pleasantville, New York.

Associate Professor Willie E. Clark, chairman of the Department of Mathematics of Agricultural, Mechanical, and Normal College, Pine Bluff, Arkansas, died on December 4, 1954.

Captain C. E. Dimick, United States Coast Guard (ret.), professor and head of the Department of Mathematics of the United States Coast Guard Academy, died on January 2, 1956. He was a charter member of the Association.

Professor Emeritus C. C. Grove of City College of the City of New York died on January 12, 1956. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania on November 26, 1955. Professor T. L. Koehler, Chairman of the Section, presided. There were 87 present, including 64 members of the Association. By-Laws for the Section were adopted by vote of those present.

Officers of the Section for the year 1955–1956 are: Chairman, Professor J. C. Oxtoby, Bryn Mawr College; Secretary, Professor G. C. Webber, University of Delaware; Members at Large—for three years, Professor N. J. Fine, University of Pennsylvania; two years, Professor T. L. Koehler, Muhlenberg College; one year, Dr. Bryant Tuckerman, Institute for Advanced Study.

The following papers were presented:

1. ***A report on the Ford Foundation study on the integration of high school and college mathematics***, by Professor H. W. Brinkmann, Swarthmore College.

Professor Brinkmann reported on the present status of the School and College Study of Admission with Advanced Standing as it concerns mathematics. He gave a brief summary of the

results of the mathematics examinations given last spring and of the Conference of Mathematics Teachers held at Williams College in June. The work of the Committee on Mathematics, as well as that of the other committees, is being carried on by the College Entrance Examination Board. Examinations for advanced standing will therefore be available (beginning next May) for any candidates who wish to take advantage of them.

2. *Dedekind sums and classes of modular substitutions*, by Professor Hans Rademacher, University of Pennsylvania.

The modular matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (a, b, c, d \text{ integers}, ad - bc = 1).$$

form a group Γ. Two matrices M and M' are called similar if there exists an $L \in \Gamma$ such that

$$M' = L^{-1}ML.$$

The relation of similarity splits Γ into classes. An invariant for similarity is obviously the trace $s = a + d$. Another invariant can be defined as follows:

$$\Psi(M) = \begin{cases} \dfrac{b}{d} & \text{for } c = 0 \\ \dfrac{a+d}{c} - 12 \operatorname{sign} c \cdot s(a, |c|) - 3 \operatorname{sign}(c(a+d)) & \text{for } c \neq 0 \end{cases}$$

where $s(h, k)$ is the Dedekind sum

$$s(h, k) = \sum_{\mu=1}^{k-1} \frac{\mu}{k} \left\{ \frac{h\mu}{k} - \left[\frac{h\mu}{k} \right] - \frac{1}{2} \right\}, \qquad (h, k) = 1.$$

The invariant Ψ has the properties

$$\Psi(-M) = \Psi(M), \qquad \Psi(M^{-1}) = -\Psi(M).$$

The inhomogeneous modular group Γ^* (*i.e.* the factor group Γ/E, where E consists of two elements, the unit matrix I and $-I$) can be generated by two elements T, U with

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

such that

$$T^2 = U^3 = 1.$$

Any $M \in \Gamma^*$ has therefore a unique representation

$$M = U^{\epsilon_0} T U^{\epsilon_1} \cdots U^{\epsilon_\nu} T U^{\epsilon_{\nu+1}}$$

with $\epsilon_0, \epsilon_{\nu+1} = 0, \pm 1$, $\epsilon_j = \pm 1$, $j = 1, 2, \cdots \nu$. The paper gives then another determination of $\Psi(M)$ by means of the ϵ_i.

3. *Flexagons*, by Professor R. J. Wisner, Haverford College.

During the last few years, a number of people have shown interest in certain geometric objects usually called flexagons. Three notes have been published about two of these: the one with three and the one with six physical faces. A method of constructing flexagons with $n \geq 3$ physical faces, some properties of "regular" flexagons, a possible algebraic definition, and the use of flexagons as realizations of certain groups are shown. The results are a part of joint work by Professor C. O. Oakley and the speaker.

4. *On differential operators*, by Professor William Feller, Princeton University.

The differential operator Ω defined by $\Omega f = af'' + bf' + cf$, where $a > 0$, has the characteristic property: if f has a local minimum at x_0 and $f(x_0) = 0$ then $\Omega f(x_0) \geqq 0$. The speaker described the form of the most general operator with this property, and the advantages arising from such generalized differential operators instead of Ω.

5. *Pea soup, tripe and mathematics*, by Professor Morris Kline, New York University.

The speaker contends that the usual material taught to freshmen, namely, college algebra and trigonometry, is meaningless, unmotivated, lacking in esthetic qualities and incoherent. These deficiencies account for the usual negative student reaction to freshman mathematics. This paper also presents objections to the replacement of this conventional material by rigorous abstract mathematics such as set theory, symbolic logic, Dedekind cuts, groups and fields and the like. The recommendation made in this talk is that mathematics be tied intimately to physical problems and to its cultural setting and significance. Physical problems should provide the motivation, meaning and significance at the same time that the role of mathematics in idealizing and reasoning about physical problems is given the chief attention so that the course is definitely one in mathematics and not a part of a physical science course. Other values of mathematics that can be presented in such an approach are included.

G. C. WEBBER, *Secretary*

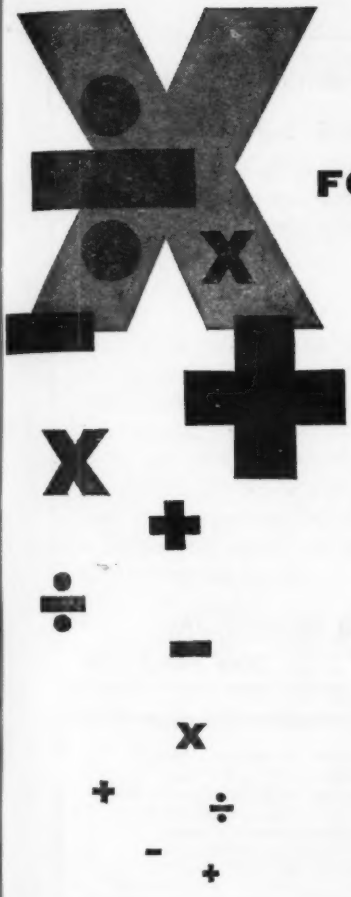
CALENDAR OF FUTURE MEETINGS

Thirty-seventh Summer Meeting, University of Washington, Seattle, Washington, August 20–21, 1956.

Fortieth Annual Meeting, University of Rochester, Rochester, New York, December 29, 1956.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Eastern Illinois State College, Charleston, May 11–12, 1956.
INDIANA, Wabash College, Crawfordsville, May 5, 1956.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, United States Naval Academy, Annapolis, Maryland, May 5, 1956.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Augsburg College, Minneapolis, May 5, 1956.
MISSOURI
NEBRASKA
NORTHEASTERN, University of Connecticut, Storrs, November 24, 1956.
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 26, 1956.
PACIFIC NORTHWEST, Oregon State College, Corvallis, June, 1957.
PHILADELPHIA
ROCKY MOUNTAIN, University of Utah, Salt Lake City, May 4–5, 1956.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Marquette University, Milwaukee, May 12, 1956.



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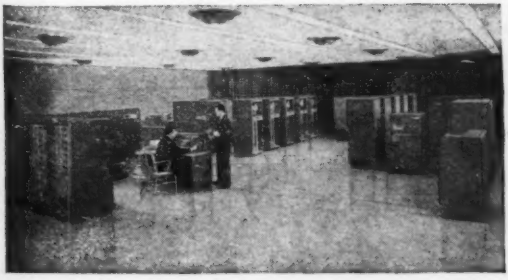

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